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# Brief Calculus

香港公開大學
THE OPEN UNIVERSITY OF HONG KONG

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# Chapter 1 Rates of Change

## 1.1 Change in discrete steps



Toward the end of the eighteenth century, a German elementary school teacher decided to keep his pupils busy by assigning them a long, boring arithmetic problem: to add up all the numbers from one to a hundred. [1] The children set to work on their slates, and the teacher lit his pipe, confident of a long break. But almost immediately, a boy named Carl Friedrich Gauss brought up his answer: 5,050.

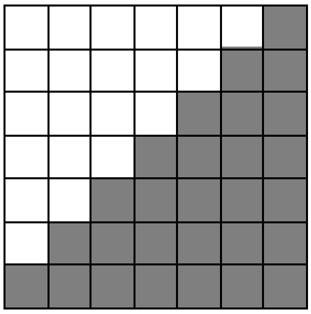

**Fig. 1.1:** Adding the numbers from 1 to 7.

Figure 1.1 suggests one way of solving this type of problem. The filled-in columns of the graph represent the numbers from 1 to 7, and adding them up means finding the area of the shaded region.

Roughly half the square is shaded in, so if we want only an approximate solution, we can simply calculate $7^2/2 = 24.5$.

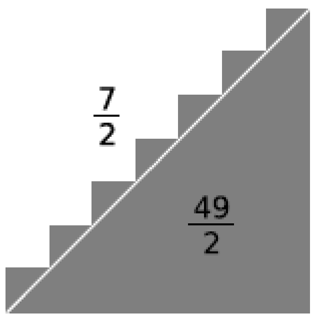


**Fig. 1.2:** A trick for finding the sum

But, as suggested in Figure 1.2, it's not much more work to get an exact result. There are seven sawteeth sticking out out above the diagonal, with a total area of $7/2$, so the total shaded area is $(7^2+7)/2 = 28$. In general, the sum of the first $n$ numbers will be$(n^2+n)/2$, which explains Gauss's result: $(100^2+100)/2 = 5,050$

1. I'm giving my own retelling of a hoary legend. We don't really know the exact problem, just that it was supposed to have been something of this flavor.

### 1.1.1 Two sides of the same coin



Problems like this come up frequently. Imagine that each household in a certain small town sends a total of one ton of garbage to the dump every year. Over time, the garbage accumulates in the dump, taking up more and more space.

**Fig. 1.3: Carl Friedrich Gauss (1777-1855)** a long time after graduating from elementary school

Let's label the years as $n = 1, 2, 3, \ldots$, and let the function [2] $x(n)$ represent the amount of garbage that has accumulated by the end of year $n$. If the population is constant, say 13 households, then garbage accumulates at a constant rate, and we have $x(n) = 13n$.

But maybe the town's population is growing. If the population starts out as 1 household in year 1, and then grows to 2 in year 2, and soon, then we have the same kind of problem that the young Gauss solved. After 100 years, the accumulated amount of garbage will be 5,050 tons. The pile of refuse grows more quickly every year; the rate of change of $x$ is not constant. Tabulating the examples we've done so far, we have this:

| rate of change | Accumulated result |
|---|---|
| 13 | $13n$ |
| $n$ | $(n^2 + n)/2$ |

The rate of change of the function $x$ can be notated as $\dot{x}$ . Given the function $\dot{x}$ , we can always determine the function $x$ for any value of $n$ by doing a running sum.

Likewise, if we know $x$, we can determine $\dot{x}$ by subtraction. In the example where $x = 13n$, we can find $\dot{x} = x(n) - x(n-1) = 13n - 13(n-1) = 13$. Or if we knew that the accumulated amount of garbage was given by $(n^2 + n)/2$, we could calculate the town's population like this:

2. Recall that when x is a function, the notation x(n) means the output of the function when the input is n. It doesn't represent multiplication of a number x by a number n.

$$\frac{n^2+n}{2} - \frac{(n-1)^2+(n-1)}{2}$$
$$= \frac{n^2+n-(n^2-2n+1+n-1)}{2}$$
$$= n$$

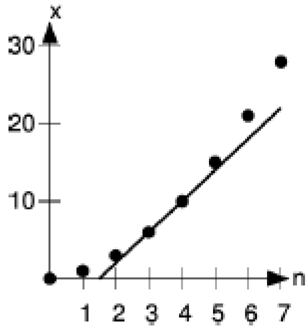


**Fig. 1.4:**

**$\dot{x}$ is the slope of x.**

The graphical interpretation of this is shown in Figure 1.4: on a graph of $x = (n^2+n)/2$, the slope of the line connecting two successive points is the value of the function $\dot{x}$ .

In other words, the functions $x$ and $\dot{x}$ are like different sides of the same coin. If you know one, you can find the other | with two caveats.

First, we've been assuming implicitly that the function $x$ starts out at $x(0) = 0$. That might not be true in general. For instance, if we're adding water to a reservoir over a certain period of time, the reservoir probably didn't start out completely empty. Thus, if we know $\dot{x}$ , we can't find out everything about $x$ without some further information: the starting value of $x$. If someone tells you $\dot{x} = 13$, you can't conclude $x = 13n$, but only $x = 13n + c$, where $c$ is some constant. There's no such ambiguity if you're going the opposite way, from $x$ to $\dot{x} = 13$ . Even if $x(0) \neq 0$, we still have $\dot{x} = 13n + c - [13(n-1) + c] = 13$.

Second, it may be difficult, or even impossible, to find a *formula* for the answer when we want to determine the running sum $x$ given a formula for the rate of change $\dot{x}$. Gauss had a flash of insight that led him to the result $(n^2+n)/2$, but in general we might only be able to use a computer spreadsheet to calculate a number for the running sum, rather than an equation that would be valid for all values of $n$.

## 1.1.2 Some guesses



Even though we lack Gauss's genius, we can recognize certain patterns. One pattern is that if $\dot{x}$ is a function that gets bigger and bigger, it seems like $x$ will be a function that grows even faster than $\dot{x}$ . In the example of $\dot{x} = n$ and $\dot{x} = (n^2+n)/2$, consider what happens for a large value of n, like 100. At this value of $n$, $\dot{x} = 100$, which is pretty big, but even without pawing around for a calculator, we know that $x$ is going to turn out really *really* big. Since $n$ is large, $n^2$ is quite a bit bigger than $n$, so roughly speaking, we can approximate $x \approx n^2/2 = 5,000$. 100 may be a big number, but 5,000 is a lot bigger. Continuing in this way, for $n = 1000$ we have $\dot{x} = 1000$, but $x \approx 500,000$ --- now $x$ has far outstripped $\dot{x}$ . This can be a fun game to play with a calculator: look at which functions grow the fastest. For instance, your calculator might have an $x^2$ button, an $e^x$ button, and a button for $x!$ (the factorial function, defined as $x! = 1 \cdot 2 \cdot ... \cdot x, \text{e.g.}, 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$). You'll find that $50^2$ is pretty big, but $e^{50}$ is incomparably greater, and $50!$ Is so big that it causes an error.

All the $x$ and $\dot{x}$ functions we've seen so far have been polynomials. If $x$ is a polynomial, then of course we can find a polynomial for $\dot{x}$ as well, because if $x$ is a polynomial, then $x(n) - x(n-1)$ will be one too. It also looks like every polynomial we could choose for $\dot{x}$ might also correspond to an $x$ that's a polynomial. And not only that, but it looks as though there's a pattern in the power of $n$. Suppose $x$ is a polynomial, and the highest power of $n$ it contains is a certain number - the "order" of the polynomial. Then $\dot{x}$ is a polynomial of that order minus one. Again, it's fairly easy to prove this going one way, passing from $x$ to $\dot{x}$ , but more difficult to prove the opposite relationship: that if $\dot{x}$ is a polynomial of a certain order, then $x$ must be a polynomial with an order that's greater by one.

We'd imagine, then, that the running sum of $\dot{x} = n^2$ would be a polynomial of order 3. If we calculate $x(100) = 1^2 + 2^2 + ... + 100^2$ on a computer spreadsheet, we get 338,350, which looks suspiciously close to 1,000,000/3. It looks like $n^3/3 + ...$, where the dots represent terms involving lower powers of $n$ such as $n^2$. The fact that the coefficient of the $n^3$ term is 1/3 is proved in Problem 1.21 (Page 24).

## Example

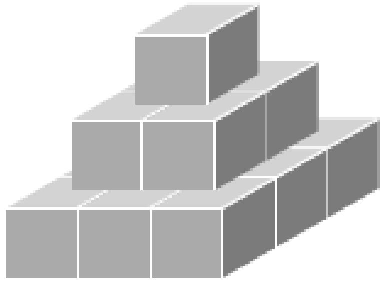

**Fig. 1.5: A pyramid with a volume of**

$$1^2 + 2^2 + 3^2$$

Figure 1.5 shows a pyramid consisting of a single cubical block on top, supported by a $2 \times 2$ layer, supported in turn by a $3 \times 3$ layer. The total volume is $1^2 + 2^2 + 3^2$, in units of the volume of a single block.

Generalizing to the sum $x(n) = 1^2 + 2^2 + ... + n^2$ ,and applying the result of the preceding paragraph, we find that the volume of such a pyramid is approximately $(1/3)Ah$, where $A = n^2$ is the area of the base and $h = n$ is the height.

When $n$ is very large, we can get as good an approximation as we like to a smooth-sided pyramid, and the error incurred in $x(n) \approx (1/3)n^3 + ...$ by omitting the lower-order terms ... can be made as small as desired.

We therefore conclude that the volume is *exactly* $(1/3)Ah$ for a smooth sided pyramid with these proportions.

This is a special case of a theorem first proved by Euclid (propositions XII-6 and XII-7) two thousand years before calculus was invented.

## 1.2 Continuous change



**Fig. 1.6: Isaac Newton (1643- 1727)**

Did you notice that I sneaked something past you in the example of water filling up a reservoir? The $x$ and $\dot{x}$ functions I've been using as examples have all been functions defined on the integers, so they represent change that happens in discrete steps, but the flow of water into a reservoir is smooth and continuous. Or is it? Water is made out of molecules, after all. It's just that water molecules are so small that we don't notice them as individuals. Figure 1.7 shows a graph that is discrete, but almost appears continuous because the scale has been chosen so that the points blend together visually.

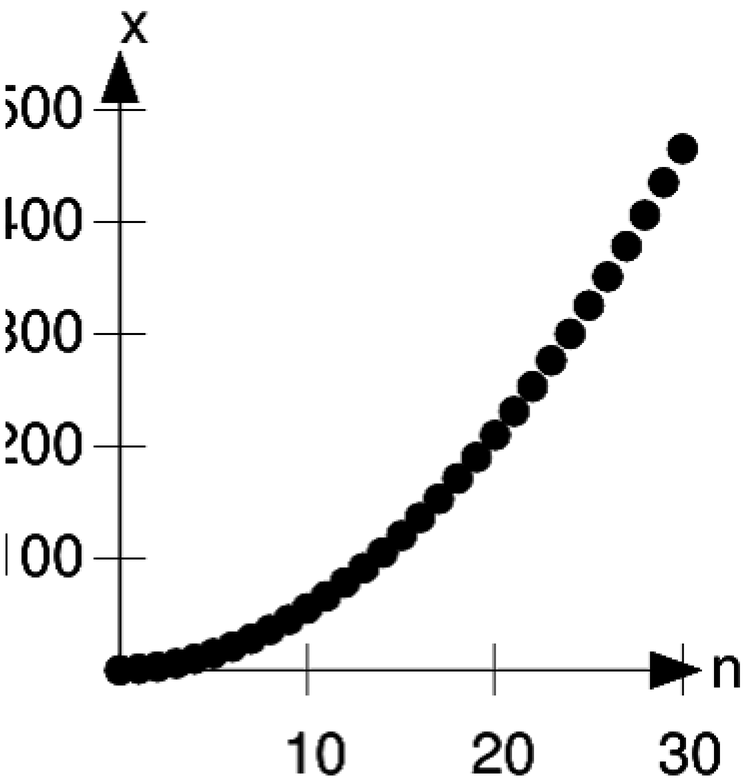


**Fig. 1.7:** On this scale, the graph of

$(n^2 + n)/2$ **appears almost continuous.**

The physicist Isaac Newton started thinking along these lines in the 1660's, and figured out ways of analyzing $x$ and $\dot{x}$ functions that were truly continuous. The notation $\dot{x}$ is due to him (and he only used it for continuous functions). Because he was dealing with the continuous flow of change, he called his new set of mathematical techniques the method of *fluxions*, but nowadays it's known as the calculus.

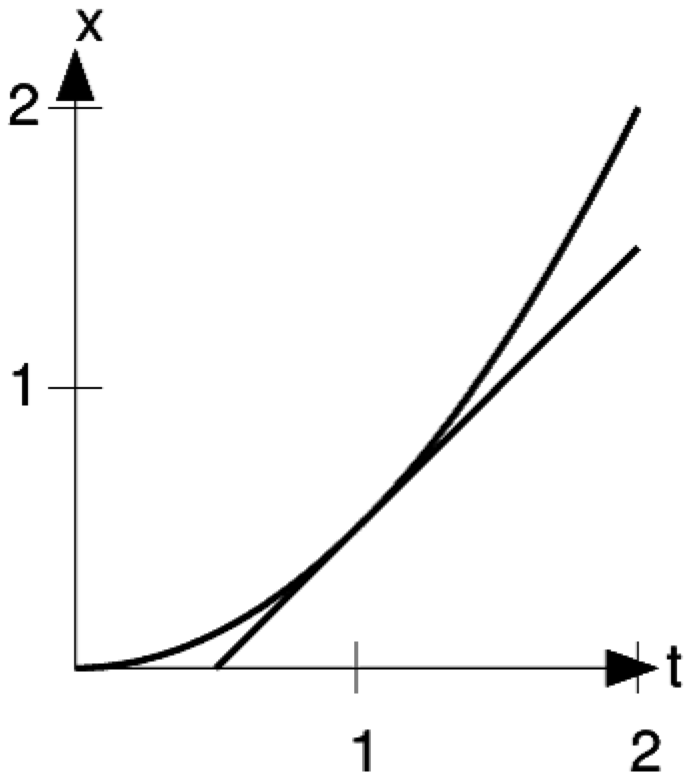


**Fig. 1.8:** The function

$x(t) = t^2/2$,**and its tangent line at the point (1, 1=2).**

Newton was a physicist, and he needed to invent the calculus as part of his study of how objects move. If an object is moving in one dimension, we can specify its position with a variable $x$, and $x$ will then be a function of time, $t$. The rate of change of its position, $\dot{x}$ , is its speed, or velocity. Earlier experiments by Galileo had established that when a ball rolled down a slope, its position was proportional to $t^2$, so Newton inferred that a graph like Figure 1.8 would be typical for any object moving under the influence of a constant force. (It could be $7t^2$, or $t^2/42$, or anything else proportional to $t^2$, depending on the force acting on the object and the object's mass.)

Because the functions are continuous, not discrete, we can no longer define the relationship between $x$ and $\dot{x}$ by saying $x$ is a running sum of $\dot{x}$ 's, or that x_ is the difference between two successive $x$'s. But we already found a geometrical relationship between the two functions in the discrete case, and that can serve as our definition for the continuous case: $x$ is the area under the graph of $\dot{x}$ , or, if you like, $\dot{x}$ is the slope of the graph of $x$. For now we'll concentrate on the slope idea.

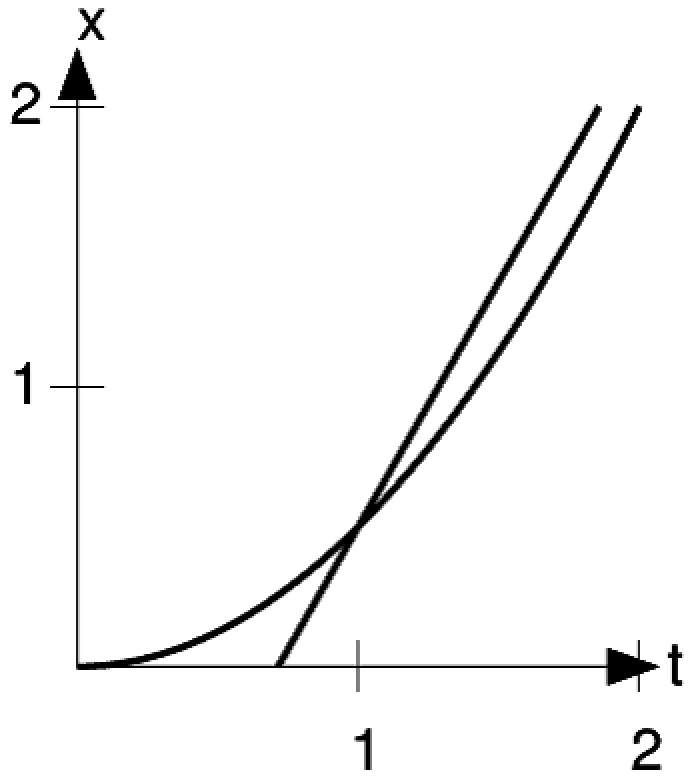


**Fig. 1.9:** This line isn’t a tangent line: it crosses the graph.

This definition is still a little vague, because we haven't defined what we mean by the "slope" of a curving graph. For a discrete graph like Figure 1.4, we could define it as the slope of the line drawn between neighboring points. Visually, it's clear that the continuous version of this is something like the line drawn in Figure 1.8. This is referred to as the tangent line.

We still need to convert this intuitive idea of a tangent line into a formal definition. In a typical example like figure h, the tangent line can be defined as the line that touches the graph at a certain point, but, unlike the line in Figure 1.9, doesn't cut across the graph at that point. [3] By measuring with a ruler on Figure 1.8, we find that the slope is very close to 1, so evidently $\dot{x}(1) = 1$. To prove this, we construct the function representing the line: $l(t) = t - 1/2$. We want to prove that this line doesn't cross the graph of $x(t) = t^2/2$. The difference between the two functions, $x - l$, is the polynomial $t^2/2 - t + 1/2$, and this polynomial will be zero for any value of $t$ where the line touches or crosses the curve. We can use the quadratic formula to _nd these points, and the result is that there is only one of them, which is $t = 1$. Since $x - l$ is positive for at least some points to the left and right of $t = 1$, and it only equals zero at $t = 1$, it must never be negative, which means that the line always lies below the curve, never crossing it.

3. In the case where the original graph is itself a line, the tangent line simply co- incides with the graph, and this also sat- ises the denition, because the tangent line doesn't cut across the graph; it lies on top of it. There is one other excep- tional case, called a point of in ection, which we won't worry about right now. For a more complicated defiition that correctly handles all the cases, see Detours (Page 169).

### 1.2.1 A derivative



That proves that $\dot{x}(1)=1$, but it was a lot of work, and we don't want to do that much work to evaluate $\dot{x}$ at every value of $t$. There's a way to avoid all that, and find a formula for $\dot{x}$ . Compare Figure 1.8 and Figure 1.10. They're both graphs of the same function, and they both look the same. What's different? The only difference is the scales: in Figure 1.10, the $t$ axis has been shrunk by a factor of 2, and the $x$ axis by a factor of 4. The graph looks the same, because doubling $t$ quadruples $t^2/2$. The tangent line here is the tangent line at $t=2$, not $t=1$, and although it looks like the same line as the one in Figure 1.8, it isn't, because the scales are different. The line in Figure 1.8 had a slope of rise/run = 1/1 = 1, but this one's slope is 4/2 = 2. That means $\dot{x}(2)=2$. In general, this scaling argument shows that $\dot{x}(t)=t$ for any $t$.

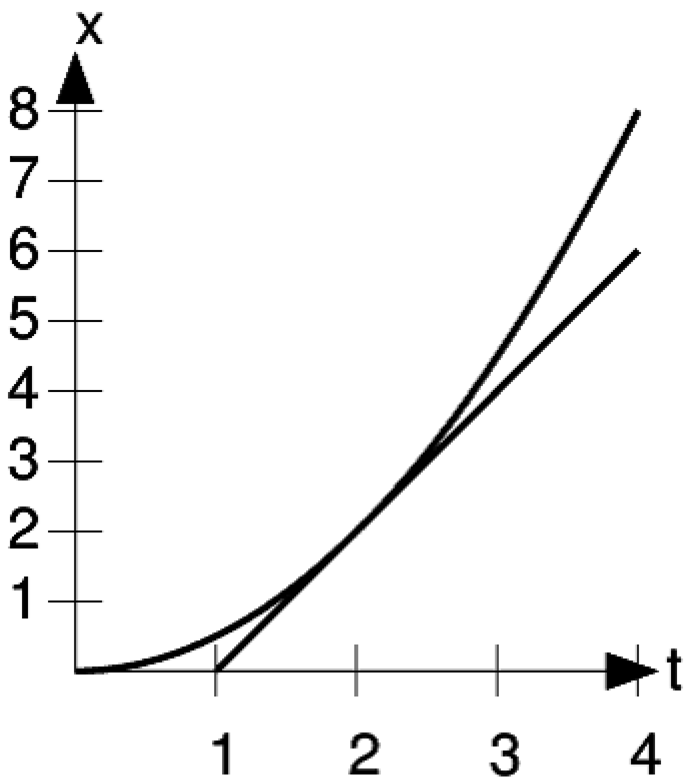


**Fig. 1.10:** The function

$t^2/2$ again. How is this different from Figure 1.8?

This is called *differentiating*: finding a formula for the function $\dot{x}$ , given a formula for the function $x$. The term comes from the idea that for a discrete function, the slope is the difference between two successive values of the function.

The function $\dot{x}$ is referred to as the *derivative* of the function $x$, and the art of differentiating is differential calculus. The opposite process, computing a formula for $x$ when given $\dot{x}$, is called integrating, and makes up the field of integral calculus; this terminology is based on the idea that computing a running sum is like putting together (integrating) many little pieces.

Note the similarity between this result for continuous functions,

$$x = t^2/2 \quad \dot{x} = t$$

and our earlier result for discrete ones,

$$x = (n^2 + n)/2 \quad \dot{x} = n$$

The similarity is no coincidence. A continuous function is just a smoothed-out version of a discrete one. For instance, the continuous version of the staircase function shown in Figure 1.2 would simply be a triangle without the saw teeth sticking out; the area of those ugly sawteeth is what's represented by the $n/2$ term in the discrete result $x = (n^2 + n)/2$, which is the only thing that makes it different from the continuous result $x = t^2/2$.

### 1.2.2 Properties of the derivative



It follows immediately from the definition of the derivative that multiplying a function by a constant multiplies its derivative by the same constant, so for example since we know that the derivative of $t^2/2$ is $t$, we can immediately tell that the derivative of $t^2$ is $2t$, and the derivative of $t^2/17$ is $2t/17$.

Also, if we add two functions, their derivatives add. To give a good example of this, we need to have another function that we can differentiate, one that isn't just some multiple of $t^2$. An easy one is $t$: the derivative of $t$ is 1, since the graph of $x = t$ is a line with a slope of 1, and the tangent line lies right on top of the original line.

**Example**

The derivative of $5t^2 + 2t$ is the derivative of $5t^2$ plus the derivative of $2t$, since derivatives add. The derivative of $5t^2$ is 5 times the derivative of $t^2$, and the derivative of $2t$ is 2 times the derivative of $t$, so putting everything together, we find that the derivative of $5t^2 + 2t$ is $(5)(2t) + (2)(1) = 10t + 2$. The derivative of a constant is zero, since a constant function's graph is a horizontal line, with a slope of zero. We now know enough to differentiate any second order polynomial.

## Example

An insect pest from the United States is inadvertently released in a village in rural China. The pests spread outward at a rate of $s$ kilometers per year, forming a widening circle of contagion. Find the number of square kilometers per year that become newly infested. Check that the units of the result make sense. Interpret the result.

Let $t$ be the time, in years, since the pest was introduced. The radius of the circle is $r = st$, and its area is $a = \pi r^2 = \pi(st)^2$. To make this look like a polynomial, we have to rewrite it as $a = (\pi s^2)t^2$. The derivative is

$$\dot{a} = (\pi s^2)(2t)$$
$$\dot{a} = (2\pi s^2)t$$

The units of $s$ are km/year, so squaring it gives $\mathrm{km}^2/\mathrm{year}^2$. The 2 and the $\pi$ are unitless, and multiplying by $t$ gives units of $\mathrm{km}^2/\mathrm{year}$, which is what we expect for $\dot{a}$, since it represents the number of square kilometers per year that become infested.

Interpreting the result, we notice a couple of things. First, the rate of infestation isn't constant; it's proportional to $t$, so people might not pay so much attention at first, but later on the effort required to combat the problem will grow more and more quickly. Second, we notice that the result is proportional to $s^2$. This suggests that anything that could be done to reduce $s$ would be very helpful. For instance, a measure that cut $s$ in half would reduce $\dot{a}$ by a factor of four.

### 1.2.3 Higher-order polynomials



So far, we have the following results for polynomials up to order

| function | derivative |
|---|---|
| 1 | 0 |
| t | 1 |
| $t^2$ | 2t |

Interpreting 1 as $t^0$, we detect what seems to be a general rule, which is that the derivative of $t^k$ is $kt^{k-1}$.
The proof is straightforward but not very illuminating if carried out with the methods developed in this chapter, so I've relegated it to Derivatives of polynomials (Page 169).

It can be proved much more easily using the methods of To infinity — and beyond! (Page 25).

## Example

If $x = 2t^7 - 4t + 1$, find $\dot{x}$.

This is similar to Example (Page 10), the only difference being that we can now handle higher powers of $t$. The derivative of $t^7$ is $7t^6$, so we have

$$\begin{aligned} \dot{x} &= (2)(7t^6) + (-4)(1) + 0 \\ &= 14t^6 - 4 \end{aligned}$$

## Example

Calculate $3^{-1}$ and $3.01^{-1}$. Does this seem consistent with a conjecture that the rule for differentiating $t^k$ holds for k < 0?

We have $3^{-1} \approx 0.33333$ and $3^{-1} \approx 0.332223$, the difference being $-1.1 \times 10^{-3}$. This suggests that the graph of $x = 1/t$ has a tangent line at $t = 3$ with a slope of about

$$\frac{-1.1 \times 10^{-3}}{0.01} = -0.11$$

If the rule for differentiating $t^k$ were to hold, then we would have $\dot{x} = t^{-2}$, and evaluating this at x = 3 would give −1/9, which is indeed about −0.11. Yes, the rule does appear to hold for negative $k$, although this numerical check does not constitute a proof. A proof is given in Example (Page 29).

### 1.2.4 The second derivative



I described how Galileo and Newton found that an object subject to an external force, starting from rest, would have a velocity $\dot{x}$ that was proportional to $t$, and a position $x$ that varied like $t^2$. The proportionality constant for the velocity is called the acceleration, $a$, so that $\dot{x} = at$ and $x = at^2/2$. For example, a sports car accelerating from a stop sign would have a large acceleration, and its velocity $at$ at a given time would therefore be a large number. The acceleration can be thought of as the derivative of the derivative of $x$, written $\ddot{x}$, with two dots. In our example, $\ddot{x} = a$. In general, the acceleration doesn't need to be constant. For example, the sports car will eventually have to stop accelerating, perhaps because the backward force of air friction becomes as great as the force pushing it forward. The total force acting on the car would then be zero, and the car would continue in motion at a constant speed.

## Example

Suppose the pilot of a blimp has just turned on the motor that runs its propeller, and the propeller is spinning up. The resulting force on the blimp is therefore increasing steadily, and let's say that this causes the blimp to have an acceleration $\ddot{x} = 3t$, which increases steadily with time. We want to find the blimp's velocity and position as functions of time.

For the velocity, we need a polynomial whose derivative is $3t$. We know that the derivative of $t^2$ is $2t$, so we need to use a function that's bigger by a factor of $3/2 : \dot{x} = (3/2)t^2$. In fact, we could add any constant to this, and make it $\dot{x} = (3/2)t^2 + 14$, for example, where the 14 would represent the blimp's initial velocity. But since the blimp has been sitting dead in the air until the motor started working, we can assume the initial velocity was zero. Remember, any time you're working backwards like this to find a function whose derivative is some other function (integrating, in other words), there is the possibility of adding on a constant like this.

Finally, for the position, we need something whose derivative is $(3/2)t^2$. The derivative of $t^3$ would be $3t^2$, so we need something half as big as this: $x = t^3/2$.

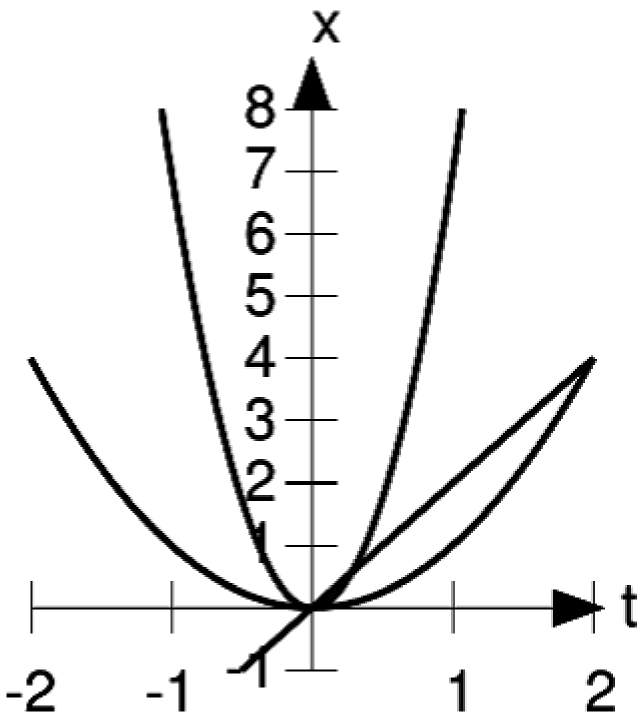


**Fig. 1.11:** The functions

$2t$, $t^2$ and $7t^2$.

The second derivative can be interpreted as a measure of the curvature of the graph, as shown in Figure 1.11. The graph of the function $x = 2t$ is a line, with no curvature.

Its first derivative is 2, and its second derivative is zero. The function $t^2$ has a second derivative of 2, and the more tightly curved function $7t^2$ has a bigger second derivative,14.

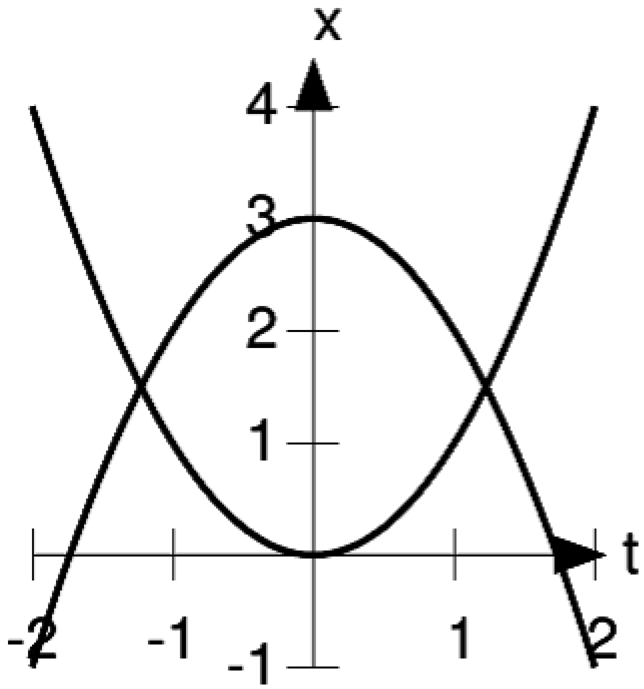


**Fig. 1.12:** The functions

$t^2$ and $3 - t^2$

Positive and negative signs of the second derivative indicate concavity. In Figure 1.12, the function $t^2$ is like a cup with its mouth pointing up. We say that it's "concave up," and this corresponds to its positive second derivative. The function $3 - t^2$, with a second derivative less than zero, is concave down. Another way of saying it is that if you're driving along a road shaped like $t^2$, going in the direction of increasing $t$, then your steering wheel is turned to the left, whereas on a road shaped like $3 - t^2$ it's turned to the right.

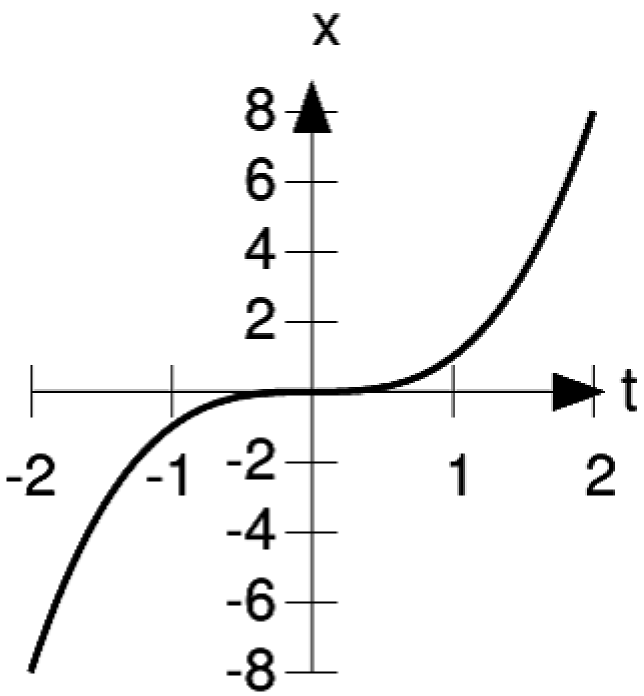


**Fig. 1.13: The functions**

$t^3$ **has an inflection point at** $t = 0$

Figure 1.13 shows a third possibility. The function $t^3$ has a derivative $3t^2$, which equals zero at $t = 0$. This called a point of inflection. The concavity of the graph is down on the left, up on the right. The inflection point is where it switches from one concavity to the other. In the alternative description in terms of the steering wheel, the inflection point is where your steering wheel is crossing from left to right.

## 1.3 Applications Maxima and minima



When a function goes up and then smoothly turns around and comes back down again, it has zero slope at the top. A place where $\dot{x} = 0$, then, could represent a place where $x$ was at a maximum. On the other hand, it could be concave up, in which case we'd have a minimum. The term extremum refers to either a maximum or a minimum.

## Example

Fred receives a mysterious e-mail tip telling him that his investment in a certain stock will have a value given by $x = -2t^4 + (6.4577 \times 10^{10})t$, where $t \geq 2005$ is the year. Should he sell at some point? If so, when?

If the value reaches a maximum at some time, then the derivative should be zero then. Taking the derivative and setting it equal to zero, we have

$$0 = -8t^3 + 6.4577 \times 10^{10}$$
$$t = \left(\frac{6.4577 \times 10^{10}}{8}\right)^{1/3}$$
$$t = \pm 2006.0$$

Obviously the solution at $t = -2006.0$ is bogus, since the stock market didn't exist four thousand years ago, and the tip only claimed the function would be valid for $t \geq 2005$.

Should Fred sell on New Year's eve of 2006?

But this could be a maximum, a minimum, or an inflection point. Fred definitely does not want to sell at $t = 2006$ if it's a minimum! To check which of the three possibilities hold, Fred takes the second derivative:

$$\ddot{x} = -24t^2$$

Plugging in $t = -2006.0$, we find that the second derivative is negative at that time, so it is indeed a maximum.

Implicit in this whole discussion was the assumption that the maximum or minimum occurred where the function was smooth. There are some other possibilities.

In Figure 1.14, the function's minimum occurs at an end-point of its domain.

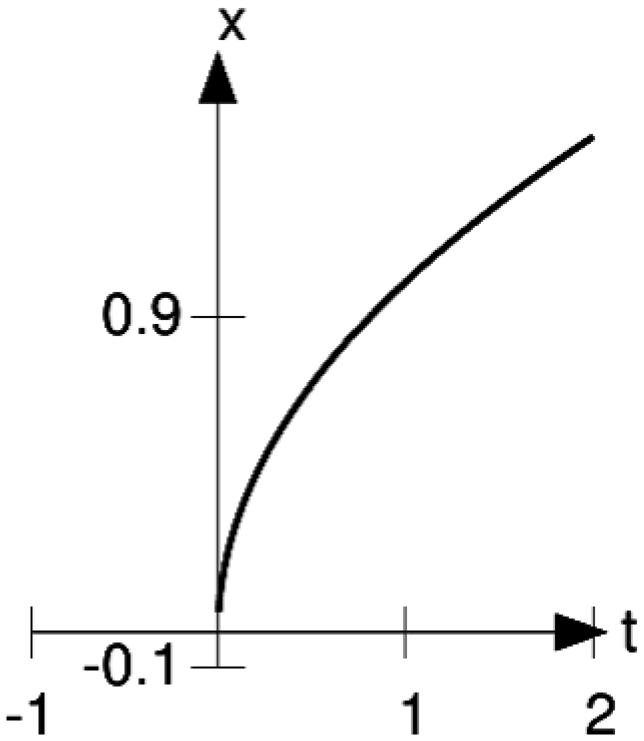


**Fig. 1.14:** The function

$x = \sqrt{t}$ **has a minimum at** $t = 0$**, which is not a place where** $\dot{x} = 0$**. This point is the edge of the function's domain.**

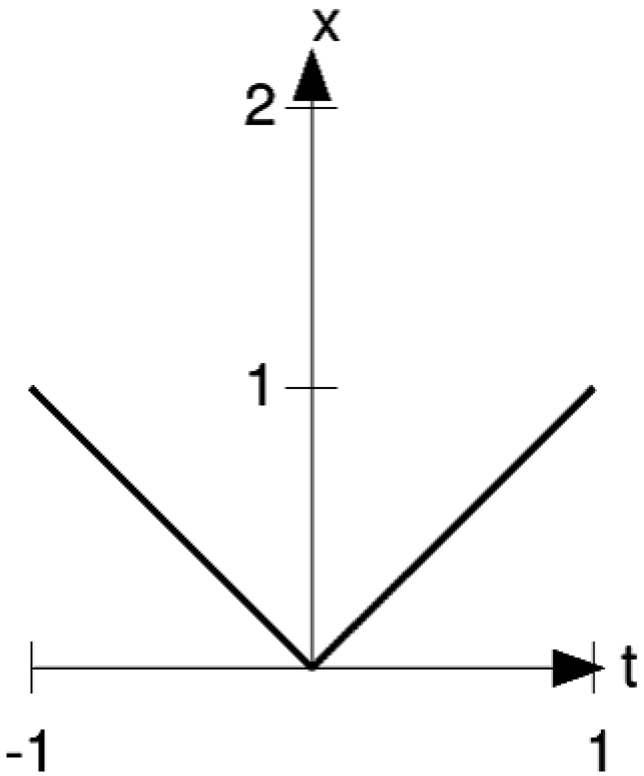


**Fig. 1.15:** The function
$x = |t|$ **has a minimum at** $t = 0$**, which is not a place where** $\dot{x} = 0$**. This is a point where the function isn't differentiable.**

Another possibility is that the function can have a minimum or maximum at some point where its derivative isn't well defined. Figure 1.15 shows such a situation. There is a kink in the function at $t = 0$, so a wide variety of lines could be placed through the graph there, all with different slopes and all staying on one side of the graph. There is no uniquely defined tangent line, so the derivative is undefined.

## Example

Rancher Rick has a length of cyclone fence $L$ with which to enclose a rectangular pasture. Show that he can enclose the greatest possible area by forming a square with sides of length $L/4$.

If the width and length of the rectangle are $t$ and $u$, and Rick is going to use up all his fencing material, then the perimeter of the rectangle, $2t+2u$, equals $L$, so for a given width, $t$, the length is $u=L/2-t$. The area is $a=tu=t(L/2-t)$. The function only means anything realistic for $0\le t\le L/2$, since for values of $t$ outside this region either the width or the height of the rectangle would be negative. The function $a(t)$ could therefore have a maximum either at a place where $\dot{a}=0$, or at the endpoints of the function's domain. We can eliminate the latter possibility, because the areais zero at the endpoints.

To evaluate the derivative, we first need to reexpress a as a polynomial

$$a=-t^2+\frac{L}{2}t$$

The derivative is

$$\dot{a}=-2t+\frac{L}{2}$$

Setting this equal to zero, we find $t=L/4$, as claimed. This is a maximum, not a minimum or an inflection point, because the second derivative is the constant $\ddot{a}=-2$, which is negative for all $t$, including $t=L/4$.

## 1.3.1 Propagation of errors



The Women's National Basketball Association says that balls used in its games should have a radius of 11.6 cm, with an allowable range of error of plus or minus 0.1 cm (one millimeter). How accurately can we determine the ball's volume?

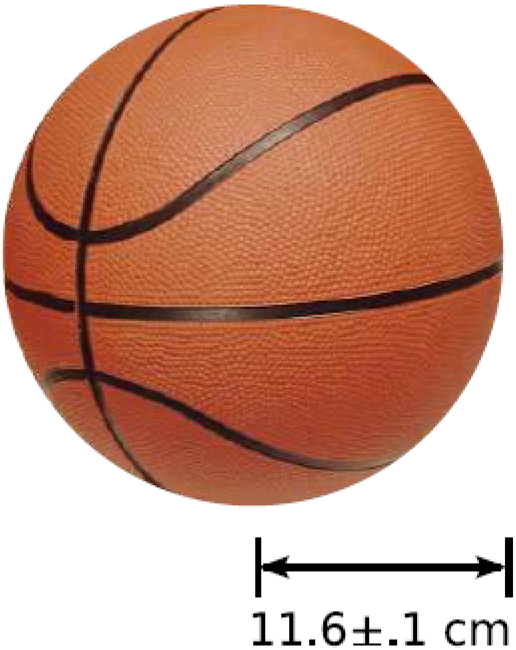


**Fig. 1.16:** How accurately can we determine the ball's volume?

The equation for the volume of a sphere gives $V = (4/3)\pi r^3 = 6538\mathrm{cm}^3$ (about six and a half liters). We have a function $V(r)$, and we want to know how much of an effect will be produced on the function's output $V$ if its input $r$ is changed by a certain small amount. Since the amount by which $r$ can be changed is small compared to $r$, it's reasonable to take the tangent line as an approximation to the actual graph. The slope of the tangent line is the derivative of $V$ , which is $4\pi r^2$. (This is the ball's surface area.) Setting (slope) = (rise)/(run) and solving for the rise, which represents the change in $V$ , we find that it could be off by as much as $(4\pi r^2)(0.1\mathrm{cm}) = 170\mathrm{cm}^3$. The volume of the ball can therefore be expressed as $6500 \pm 170\mathrm{cm}^3$, where the original figure of 6538 has been rounded off to the nearest hundred in order to avoid creating the impression that the 3 and the 8 actually mean anything | they clearly don't, since the possible error is out in the hundreds' place.

This calculation is an example of a very common situation that occurs in the sciences, and even in everyday life, in which we base a calculation on a number that has some range of uncertainty in it, causing a corresponding range of uncertainty in the _nal result. This is called propagation of errors. The idea is that the derivative expresses how sensitive the function's output is to its input.

The example of the basketball could also have been handled without calculus, simply by recalculating the volume using a radius that was raised from 11.6 to 11.7 cm, and finding the difference between the two volumes. Understanding it in terms of calculus, however, gives us a different way of getting at the same ideas, and often allows us to understand more deeply what's going on. For example, we noticed in passing that the derivative of the volume was simply the surface area of the ball, which provides a nice geometric visualization. We can imagine inflating the ball so that its radius is increased by a millimeter. The amount of added volume equals the surface area of the ball multiplied by one millimeter, just as the amount of volume added to the world's oceans by global warming equals the oceans' surface area multiplied by the added depth.

For an example of an insight that we would have missed if we hadn't applied calculus, consider how much error is incurred in the measurement of the width of a book if the ruler is placed on the book at a slightly incorrect angle, so that it doesn't form an angle of exactly 90 degrees with spine. The measurement has its minimum (and correct) value if the ruler is placed at exactly 90 degrees. Since the function has a minimum at this angle, its derivative is zero. That means that we expect essentially no error in the measurement if the ruler's angle is just a tiny bit o_. This gives us the insight that it's not worth fiddling excessively over the angle in this measurement. Other sources of error will be more important. For example, is the book a uniform rectangle? Are we using the worn end of the ruler as its zero, rather than letting the ruler hang over both sides of the book and subtracting the two measurements?

# 1.4 Problems

## 1.4.1 Problem 1.1



Graph the function $t^2$ in the neighborhood of $t = 3$, draw a tangent line, and use its slope to verify that the derivative equals $2t$ at this point

Solutions for chapter 1 (Page 189)

## 1.4.2 Problem 1.2



Graph the function $\sin\ e^t$ in the neighborhood of $t = 0$, draw a tangent line, and use its slope to estimate the derivative. Answer: 0.5403023058. (You will of course not get an answer this precise using this technique.) .

Solutions for chapter 1 (Page 189)

## 1.4.3 Problem 1.3



Differentiate the following functions with respect to$t : 1, 7, t, 7t, t^2, 7t^2, t^3, 7t^3$.

Solutions for chapter 1 (Page 189)

## 1.4.4 Problem 1.4



Differentiate $3t^7 - 4t^2 + 6$ with respect to $t$.

Solutions for chapter 1 (Page 189)

### 1.4.5 Problem 1.5



Differentiate $at^2 + bt + c$ with respect to $t$.

Solutions for chapter 1 (Page 189)

### 1.4.6 Problem 1.6



Find two different functions whose derivatives are the constant 3, and give a geometrical interpretation.

Solutions for chapter 1 (Page 189)

### 1.4.7 Problem 1.7



Find a function $x$ whose derivative is $\dot{x} = t^7$. In other words, integrate the given function.

Solutions for chapter 1 (Page 189)

### 1.4.8 Problem 1.8



Find a function $x$ whose derivative is $\dot{x} = 3t^7$ . In other words, integrate the given function.

Solutions for chapter 1 (Page 189)

### 1.4.9 Problem 1.9



Find a function $x$ whose derivative is $\dot{x} = 3t^7 - 4t^2 + 6$. In other words, integrate the given function.

Solutions for chapter 1 (Page 189)

## 1.4.10 Problem 1.10



Let $t$ be the time that has elapsed since the Big Bang. In that time, one would imagine that light, traveling at speed $c$, has been able to travel a maximum distance $ct$. (In fact the distance is several times more than this, because according to Einstein's theory of general relativity, space itself has been expanding while the ray of light was in transit.) The portion of the universe that we can observe would then be a sphere of radius $ct$, with volume $v = (4/3)\pi r^3 = (4/3)\pi(ct)^3$. Compute the rate $\dot{v}$ at which the volume of the observable universe is increasing, and check that your answer has the right units, as in Example (Page 11).

Solutions for chapter 1 (Page 189)

## 1.4.11 Problem 1.11



Kinetic energy is a measure of an object's quantity of motion; when you buy gasoline, the energy you're paying for will be converted into the car's kinetic energy (actually only some of it, since the engine isn't perfectly efficient). The kinetic energy of an object with mass $m$ and velocity $v$ is given by $K = (1/2)mv^2$. For a car accelerating at a steady rate, with $v = at$, find the rate $\dot{K}$ at which the engine is required to put out kinetic energy. $\dot{K}$, with units of energy over time, is known as the *power*. Check that your answer has the right units, as in Example (Page 11).

Solutions for chapter 1 (Page 189)

## 1.4.12 Problem 1.12



A metal square expands and contracts with temperature, the lengths of its sides varying according to the equation $l = (1 + \alpha T)l_o$. Find the rate of change of its surface area $a$ with respect to temperature. That is, find $\dot{a}$, where the variable with respect to which you're differentiating is the temperature, $T$. Check that your answer has the right units, as in Example (Page 11).

Solutions for chapter 1 (Page 189)

## 1.4.13 Problem 1.13



Find the second derivative of $2t^3 - t$.

Solutions for chapter 1 (Page 189)

## 1.4.14 Problem 1.14



Locate any points of infection of the function $t^3 + t^2$. Verify by graphing that the concavity of the function reverses itself at this point.

Solutions for chapter 1 (Page 189)

## 1.4.15 Problem 1.15



Let's see if the rule that the derivative of $t^k$ is $kt^{k-1}$ also works for $k < 0$. Use a graph to test one particular case, choosing one particular negative value of $k$, and one particular value of $t$. If it works, what does that tell you about the rule? If it doesn't work?.

Solutions for chapter 1 (Page 189)

## 1.4.16 Problem 1.16



Two atoms will interact via electrical forces between their protons and electrons. To put them at a distance $r$ from one another (measured from nucleus to nucleus), a certain amount of energy $E$ is required, and the minimum energy occurs when the atoms are in equilibrium, forming a molecule. Often a fairly good approximation to the energy is the Lennard-Jones expression

$$E(r) = k\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right]$$

where $k$ and $a$ are constants. Note that, as proved in To infinity — and beyond! (Page 25), the rule that the derivative of $t^k$ is $kt^{k-1}$ also works for $k < 0$. Show that there is an equilibrium at $r = a$. Verify (either by graphing or by testing the second derivative) that this is a minimum, not a maximum or a point of inflection.

Solutions for chapter 1 (Page 189)

## 1.4.17 Problem 1.17



Prove that the total number of maxima and minima possessed by a third-order polynomial is at most two.

Solutions for chapter 1 (Page 189)

## 1.4.18 Problem 1.18



Functions $f$ and $g$ are defined on the whole real line, and are differentiable everywhere. Let $s = f + g$ be their sum. In what ways, if any, are the extrema of $f$, $g$, and $s$ related?

Solutions for chapter 1 (Page 189)

## 1.4.19 Problem 1.19



Euclid proved that the volume of a pyramid equals $(1/3)bh$, where $b$ is the area of its base, and $h$ its height. A pyramidal tent without tent-poles is erected by blowing air into it under pressure. The area of the base is easy to measure accurately, because the base is nailed down, but the height fluctuates somewhat and is hard to measure accurately. If the amount of uncertainty in the measured height is plus or minus $e_h$, find the amount of possible error $e_V$ in the volume.

Solutions for chapter 1 (Page 189)

## 1.4.20 Problem 1.20



A hobbyist is going to measure the height to which her model rocket rises at the peak of its trajectory. She plans to take a digital photo from far away and then do trigonometry to determine the height, given the baseline from the launchpad to the camera and the angular height of the rocket as determined from analysis of the photo. Comment on the error incurred by the inability to snap the photo at exactly the right moment.

Solutions for chapter 1 (Page 189)

## 1.4.21 Problem 1.21



Prove, as claimed on Some guesses (Page 3), that if the sum $1^2 + 2^2 + \ldots + n^2$ is a polynomial, it must be of third order, and the coefficient of the $n^3$ term must be $1/3$.

Solutions for chapter 1 (Page 189)

# Chapter 2 To infinity — and beyond!



**Fig. 2.1: Gottfried Leibniz (1646-1716)**

Little kids readily pick up the idea of infinity. “When I grow up, I'm gonna have a million Barbies."Oh yeah? Well, I'm gonna have a billion." “Well, I'm gonna have infinity Barbies.”” So what? I'll have two infinity of them." Adults laugh, convinced that infinity, $\infty$, is the biggest number, so $2\infty$ can't be any bigger. This is the idea behind a joke in the movie Toy Story. Buzz Lightyear's slogan is “To infinity - and beyond!" We assume there *isn't* any beyond. Infinity is supposed to be the biggest there is, so by definition there can't be anything bigger, right?

## 2.1 Infinitesimals



Actually mathematicians have invented many different logical systems for working with infinity, and in most of them infinity does come in different sizes and flavors. Newton, as well as the German mathematician Leibniz who invented calculus independently, [1] had a strong intuitive idea that calculus was really about numbers that were infinitely small: infinitesimals, the opposite of infinities. For instance, consider the number $1.1^2 = 1.21$. That 2 in the first decimal place is the same 2 that appears in the expression $2t$ for the derivative of $t^2$.

(1.1, 1.21).

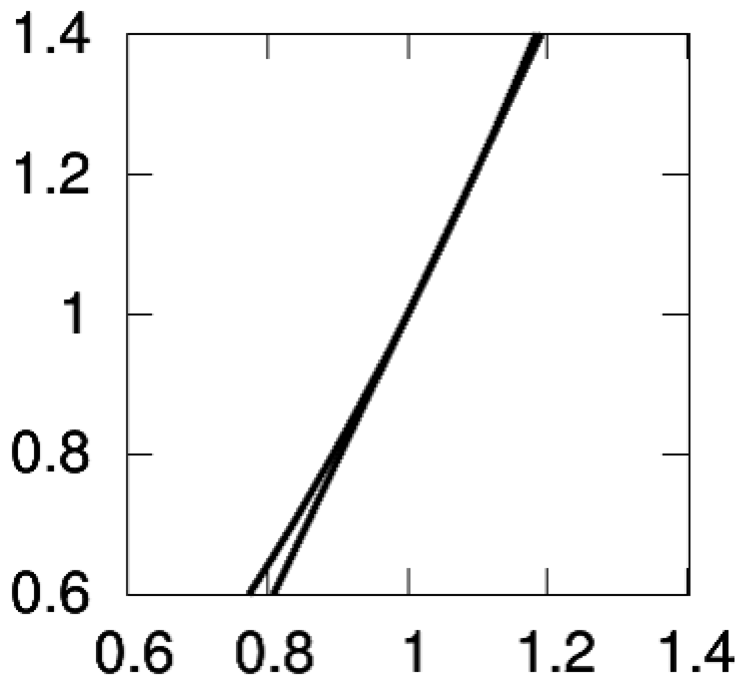


**Fig. 2.2:** A close-up view of the function

$x = t^2$, **showing the line that connects the points (1, 1) and**

Figure 2.2shows the idea visually. The line connecting the points (1, 1) and (1.1, 1.21) is almost indistinguishable from the tangent line on this scale. Its slope is (1.21 1)/(1.1 1) = 2.1, which is very close to the tangent line's slope of 2. It was a good approximation because the points were close together, separated by only 0.1 on the $t$ axis.

If we needed a better approximation, we could try calculating $1.01^2 = 1.0201$. The slope of the line connecting the points (1, 1) and (1.01, 1.0201) is 2.01, which is even closer to the slope of the tangent line.

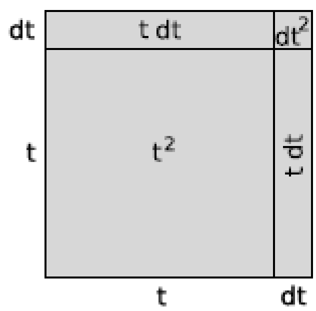


**Fig. 2.3: A geometrical interpretation of the derivative of**

1. There is some dispute over this point. Newton and his supporters claimed that Leibniz plagiarized Newton's ideas, and merely invented a new notation for them.

$t^2$

Another method of visualizing the idea is that we can interpret $x = t^2$ as the area of a square with sides of length $t$, as suggested in Figure 2.3. We increase $t$ by an infinitesimally small number $dt$. The $d$ is Leibniz's notation for a very small difference, and $dt$ is to be read as a single symbol, "dee-tee," not as a number $d$ multiplied by a number $t$. The idea is that $dt$ is smaller than any ordinary number you could imagine, but it's not zero. The area of the square is increased by $dx = 2tdt + dt^2$, which is analogous to the finite numbers 0.21 and 0.0201 we calculated earlier. Where before we divided by a finite change in $t$ such as 0.1 or 0.01, now we divide by $dt$, producing

$$\begin{aligned}\frac{dx}{dt} &= \frac{2tdt + dt^2}{dt} \\ &= 2t + dt\end{aligned}$$

for the derivative. On a graph like Figure 2.2, $dx/dt$ is the slope of the tangent line: the change in $x$ divided by the changed in $t$.

But adding an infinitesimal number $dt$ onto $2t$ doesn't really change it by any amount that's even theoretically measurable in the real world, so the answer is really $2t$. Evaluating it at $t = 1$ gives the exact result, 2, that the earlier approximate results, 2.1 and 2.01, were getting closer and closer to.

## Example

To show the power of infinitesimals and the Leibniz notation, let's prove that the derivative of $t^3$ is $3t^2$:

$$\begin{aligned}\frac{dx}{dt} &= \frac{(t + dt)^3 - t^3}{dt} \\ &= \frac{3t^2 dt + 3tdt^2 + dt^3}{dt} \\ &= 3t^2 + \ldots\end{aligned}$$

where the dots indicate infinitesimal terms that we can neglect.

This result required significant sweat and ingenuity when proved on Derivatives of polynomials (Page 169) by the methods of Rates of Change (Page 1), and not only that but the old method would have required a completely different method of proof for a function that wasn't a polynomial, whereas the new one can be applied more generally, as we'll see presently in Example (Page 29);Example (Page 29);Example (Page 30) and Example (Page 31).

It's easy to get the mistaken impression that infinitesimals exist in some remote fairyland where we can never touch them. This may be true in the same artsy-fartsy sense that we can never truly understand $\sqrt{2}$, because its decimal expansion goes on forever, and we therefore can never compute it exactly. But in practical work, that doesn't stop us from working with $\sqrt{2}$. We just approximate it as, e.g., 1.41. Infinitesimals are no more or less mysterious than irrational numbers, and in particular we can represent them concretely on a computer. If you go to `lightandmatter.com/calc/inf`, you'll find a web-based calculator called Inf, which can handle infinite and infinitesimal numbers. It has a built-in symbol, d, which represents an infinitesimally small number such as the dx's and dt's we've been handling symbolically.

Let's use Inf to verify that the derivative of $t^3$, evaluated at $t = 1$, is equal to 3, as found by plugging in to the result of Example (Page 27). The : symbol is the prompt that shows you Inf is ready to accept your typed input.

```
: ((1+d)^3-1)/d
3+3d+d^2
```

As claimed, the result is 3, or close enough to 3 that the infinitesimal error doesn't matter in real life. It might look like Inf did this example by using algebra to simplify the expression, but in fact Inf doesn't know anything about algebra. One way to see this is to use Inf to compare d with various real numbers:

```
: d<1
true
: d<0.01
true
: d<0.0000001
true
: d<0
false
```

If d were just a variable being treated according to the axioms of algebra, there would be no way to tell how it compared with other numbers without having some special information. Inf doesn't know algebra, but it does know that d is a positive number that is less than any positive real number that can be represented using decimals or scientific notation.

## Example

In Example (Page 12), we made a rough numerical check to see if the differentiation rule $t^k \to kt^{k-1}$, which was proved on Derivatives of polynomials (Page 169) for $k = 1, 2, 3, \ldots$, was also valid for $k = -1$, i.e., for the function $x = 1/t$. Let's look for an actual proof. To find a natural method of attack, let's first redo the numerical check in a slightly more suggestive form. Again approximating the derivating at $t = 3$, we have

$$\frac{dx}{dt} \approx \left(\frac{1}{3.01} - \frac{1}{3}\right)\left(\frac{1}{0.01}\right)$$

Let's apply the grade-school technique for subtracting fractions, in which we first get them over the same denominator:

$$\frac{1}{3} - \frac{1}{3.01} = \frac{3 - 3.01}{3 \times 3.01}$$

The result is

$$\begin{aligned}\frac{dx}{dt} &\approx \left(\frac{1}{3.01} - \frac{1}{3}\right)\left(\frac{1}{0.01}\right)\\ &= -\frac{1}{3 \times 3.01}\end{aligned}$$

Replacing 3 with $t$ and 0.01 with $dt$, this becomes

$$\begin{aligned}\frac{dx}{dt} &= -\frac{1}{t(t+dt)}\\ &= -t^{-2} + \ldots\end{aligned}$$

## Example

The derivative of $x = \sin\, t$, with $t$ in units of radians, is

$$\frac{dx}{dt} = \frac{\sin\,(t+dt) - \sin\, t}{dt}$$

and with the trig identity $\sin(\alpha + \beta) = \sin\,\alpha\,\cos\,\beta + \cos\,\alpha\,\sin\,\beta$, this becomes

$$= \frac{\sin\, t\,\cos\, dt + \cos\, t\,\sin\, dt - \sin\, t}{dt}$$

Applying the small-angle approximations $\sin\, u \approx u$ and $\cos\, u \approx 1$ we have

$$\begin{aligned}\frac{dx}{dt} &= \frac{\cos\, t dt}{dt} + \cdots\\ &= \cos\, t + \cdots\end{aligned}$$

where ". . . " represents the error caused by the small-angle approximations.

This is essentially all there is to the computation of the derivative, except for the remaining technical point that we haven't proved that the small-angle approximations are good enough. In Example (Page 27), when we calculated the derivative of $t^3$, the resulting expression for the quotient dx= dt came out in a form in which we could inspect the ". . . " terms and verify before discarding them that they were infinitesimal. The issue is less trivial in the present example. This point is addressed more rigorously on Details of the proof of the derivative of the sine function (Page 170)

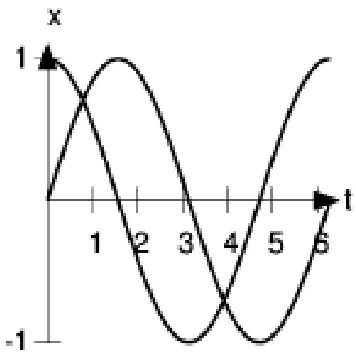


**Fig. 2.4: Graphs of $\sin\ t$, and its derivative $\cos\ t$.**

Figure 2.4 shows the graphs of the function and its derivative. Note how the two graphs correspond. At $t = 0$, the slope of sin t is at its largest, and is positive; this is where the derivative, cos t, attains its maximum positive value of 1. At $t = \pi/2$, $\sin\ t$ has reached a maximum, and has a slope of zero; $\cos\ t$ is zero here. At $t = \pi$, in the middle of the graph, sin t has its maximum negative slope, and cos t is at its most negative extreme of -1.

Physically, $\sin\ t$ could represent the position of a pendulum as it moved back and forth from left to right, and cos t would then be the pendulum's velocity.

## Example

What about the derivative of the cosine? The cosine and the sine are really the same function, shifted to the left or right by $\pi/2$. If the derivative of the sine is the same as itself, but shifted to the left by $\pi/2$, then the derivative of the cosine must be a cosine shifted to the left by $\pi/2$:

$$\begin{aligned}\frac{d\ \cos\ t}{dt} &= \cos\ (t + \pi/2)\\ &= -\sin\ t\end{aligned}$$

The next example will require a little trickery. By the end of this chapter you'll learn general techniques for cranking out any derivative cookbook-style, without having to come up with any tricks.

## Example

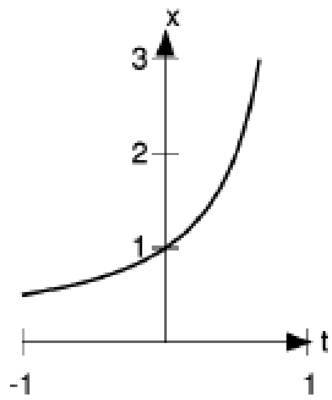


**Fig. 2.5: The function**

$x = 1/(1-t)$

Find the derivative of $1/(1-t)$, evaluated at $t = 0$.

The graph shows what the function looks like. It blows up to infinity at $t = 1$, but it's well behaved at $t = 0$, where it has a positive slope.

For insight, let's calculate some points on the curve. The point at which we're differentiating is (0, 1). If we put in a small, positive value of $t$, we can observe how much the result increases relative to 1, and this will give us an approximation to the derivative. For example, we find that at $t$ = 0.001, the function has the value 1.001001001001, and so the derivative is approximately (1.0011)/(.001-0), or about 1. We can therefore conjecture that the derivative is exactly 1, but that's not the same as proving it.

But let's take another look at that number 1.001001001001. It's clearly a repeating decimal. In other words, it appears that

$$\frac{1}{1-1/1000} = 1 + \frac{1}{1000} + \left(\frac{1}{1000}\right)^2 + \ldots$$

and we can easily verify this by multiplying both sides of the equation by 1-1/1000 and collecting like powers. This is a special case of the geometric series

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots$$

which can be derived [2] by doing synthetic division (the equivalent of long division for polynomials), or simply verified, after forming the conjecture based on the numerical example above, by multiplying both sides by 1−t.

As we'll see in Safe use of infinitesimals (Page 32), and have been implicitly assuming so far, infinitesimals obey all the same elementary laws of algebra as the real numbers, so the above derivation also holds for an infinitesimal value of $t$. We can verify the result using Inf:

```
: 1/(1-d)
1+d+d^2+d^3+d^4
```

Notice, however, that the series is truncated after the first five terms. This is similar to the truncation that happens when you ask your calculator to find $\sqrt{2}$ as a decimal.

The result for the derivative is

$$\frac{dx}{dt} = \frac{(1+dt+dt^2+...)-1}{1+dt-1}$$
$$= 1 + ...$$

## 2.2 Safe use of infinitesimals



The idea of infinitesimally small numbers has always irked purists.

**Fig. 2.6: Bishop George Berkeley (1685-1753)**

One prominent critic of the calculus was Newton's contemporary George Berkeley, the Bishop of Cloyne. Although some of his complaints are clearly wrong (he denied the possibility of the second derivative), there was clearly something to his criticism of the

2. As a technical aside, it's not necessary for our present purposes to go into the issue of how to make the most general possible definition of what is meant by a sum like this one which has an infinite number of terms; the only fact we'll need here is that the error infinite sum obtained by leaving out the ". . . " has only higher powers of t. This is taken up in more detail in Sequences and Series (Page 127) . Note that the series only gives the right answer for t < 1. E.g., for t = 1, it equals 1+1+1+:..., which, if it means anything, clearly means something infinite.

infinitesimals. He wrote sarcastically, “They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?"

Infinitesimals seemed scary, because if you mishandled them, you could prove absurd things. For example, let $du$ be an infinitesimal. Then $2du$ is also infinitesimal. Therefore both $1/du$ and $1/(2du)$ equal infinity, so $1/du = 1/(2du)$. Multiplying by $du$ on both sides, we have a proof that 1 = 1/2.

In the eighteenth century, the use of infinitesimals became like adultery: commonly practiced, but shameful to admit to in polite circles. Those who used them learned certain rules of thumb for handling them correctly. For instance, they would identify the aw in my proof of 1 = 1/2 as my assumption that there was only one size of infinity, when actually $1/du$ should be interpreted as an infinity twice as big as $1/(2du)$. The use of the symbol $\infty$ played into this trap, because the use of a single symbol for infinity implied that infinities only came in one size. However, the practitioners of infinitesimals had trouble articulating a clear set of principles for their proper use, and couldn't prove that a self-consistent system could be built around them.

By the twentieth century, when I learned calculus, a clear consensus had formed that infinite and infinitesimal numbers weren't numbers at all. A notation like $dx/dt$, my calculus teacher told me, wasn't really one number divided by another, it was merely a symbol for something called a limit,

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

where $\Delta x$ and $\Delta t$ represented finite changes. I'll give a formal definition (actually two different formal definitions) of the term "limit" in Limits (Page 67), but intuitively the concept is that we can get as good an approximation to the derivative as we like, provided that we make $\Delta t$ small enough.

That satisfied me until we got to a certain topic (implicit differentiation) in which we were encouraged to break the $dx$ away from the $dt$, leaving them on opposite sides of the equation. I buttonholed my teacher after class and asked why he was now doing what he'd told me you couldn't really do, and his response was that $dx$ and $dt$ weren't really numbers, but most of the time you could get away with treating them as if they were, and you would get the right answer in the end. *Most of the time!?* That bothered me. How was I supposed to know when it *wasn'*t "most of the time?"

**Fig. 2.7: Abraham Robinson (1918-1974)**

But unknown to me and my teacher, mathematician Abraham Robinson had already shown in the 1960's that it was possible to construct a self-consistent number system that included infinite and infinitesimal numbers. He called it the hyperreal number system, and it included the real numbers as a subset. [3]

Moreover, the rules for what you can and can't do with the hyperreals turn out to be extremely simple. Take any true statement about the real numbers. Suppose it's possible to translate it into a statement about the hyperreals in the most obvious way, simply by replacing the word "real" with the word "hyperreal." Then the translated statement is also true. This is known as the *transfer principle*.

Let's look back at my bogus proof of $1 = 1/2$ in light of this simple principle. The final step of the proof, for example, is perfectly valid: multiplying both sides of the equation by the same thing. The following statement about the real numbers is true:

For any real numbers $a$, $b$, and $c$, if $a = b$, then $ac = bc$.

This can be translated in an obvious way into a statement about the hyperreals:

For any hyperreal numbers $a$, $b$, and $c$, if $a = b$, then $ac = bc$.

However, what about the statement that both $1/du$ and $1/(2du)$ equal infinity, so they're equal to each other? This isn't the translation of a statement that's true about the reals, so there's no reason to believe it's true when applied to the hyperreals | and in fact it's false.

What the transfer principle tells us is that the real numbers as we normally think of them are not unique in obeying the ordinary rules of algebra. There are completely different systems of numbers, such as the hyperreals, that also obey them.

3. The main text of this book treats iinfinitesimal with the minimum fuss necessary in order to avoid the common goofs. More detailed discussions are of- ten relegated to the back of the book, as in Example (Page 29). The reader who wants to learn even more about the hyperreal system should consult the list of further reading on References and Further Reading (Page 219).

How, then, are the hyperreals even different from the reals, if everything that's true of one is true of the other? But recall that the transfer principle doesn't guarantee that every statement about the reals is also true of the hyperreals. It only works if the statement about the reals can be translated into a statement about the hyperreals in the most simple, straightforward way imaginable, simply by replacing the word “real" with the word “hyperreal." Here's an example of a true statement about the reals that can't be translated in this way:

For any real number $a$, there is an integer $n$ that is greater than $a$.

This one can't be translated so simplemindedly, because it refers to a subset of the reals called the integers. It might be possible to translate it somehow, but it would require some insight into the correct way to translate that word “integer." The transfer principle doesn't apply to this statement, which indeed is false for the hyperreals, because the hyperreals contain infinite numbers that are greater than all the integers. In fact, the contradiction of this statement can be taken as a definition of what makes the hyperreals special, and different from the reals: we assume that there is at least one hyperreal number, $H$, which is greater than all the integers.

As an analogy from everyday life, consider the following statements about the student body of the high school I attended:

1. Every student at my high school had two eyes and a face.
2. Every student at my high school who was on the football team was a jerk.

Let's try to translate these into statements about the population of California in general. The student body of my high school is like the set of real numbers, and the present-day population of Californiais like the hyperreals. Statement 1 can be translated mindlessly into a statement that every Californian has two eyes and a face; we simply substitute “every Californian" for “every student at my high school." But statement 2 isn't so easy, because it refers to the subset of students who were on the football team, and it's not obvious what the corresponding subset of Californians would be. Would it include everybody who played high school, college, or pro football? Maybe it shouldn't include the pros, because they belong to an organization covering a region bigger than California. Statement 2 is the kind of statement that the transfer principle doesn't apply to. [4]

4. 4For a slightly more precise and formal statement of the transfer principle, see Formal statement of the transfer principle (Page 172).

## Example

As a nontrivial example of how to apply the transfer principle, let's consider how to handle expressions like the one that occurred when we wanted to differentiate $t^2$ using infinitesimals:

$$\frac{d(t^2)}{dt} = 2t + dt$$

I argued earlier that $2t + dt$ is so close to $2t$ that for all practical purposes, the answer is really $2t$. But is it really valid in general to say that $2t + dt$ is the same hyperreal number as $2t$? No. We can apply the transfer principle to the following statement about the reals:

For any real numbers $a$ and $b$, with $b \neq 0$, $a + b \neq a$.

Since dt isn't zero, $st + dt \neq 2t$.

More generally, Example (Page 36) leads us to visualize every number as being surrounded by a "halo" of numbers that don't equal it, but differ from it by only an infinitesimal amount. Just as a magnifying glass would allow you to see the fleas on a dog, you would need an infinitely strong microscope to see this halo. This is similar to the idea that every integer is surrounded by a bunch of fractions that would round off to that integer. We can define the *standard part* of a finite hyperreal number, which means the unique real number that differs from it infinitesimally. For instance, the standard part of $2t + dt$, notated $st(2t + dt)$, equals $2t$. The derivative of a function should actually be defined as the standard part of $dx/dt$, but we often write $dx/dt$ to mean the derivative, and don't worry about the distinction.

One of the things Bishop Berkeley disliked about infinitesimals was the idea that they existed in a kind of hierarchy, with $dt^2$ being not just infinitesimally small, but infinitesimally small compared to the infinitesimal $dt$. If $dt$ is the flea on a dog, then $dt^2$ is a submicroscopic flea that lives on the flea, as in Swift's doggerel: "Big fleas have little fleas/ On their backs to ride 'em,/ and little fleas have lesser fleas,/And so, ad infinitum." Berkeley's criticism was off the mark here: there is such a hierarchy. Our basic assumption about the hyperreals was that they contain at least one infinite number, $H$, which is bigger than all the integers. If this is true, then $1/H$ must be less than $1/2$, less than $1/100$, less then $1/1,000,000$ - less than $1/n$ for any integer $n$. Therefore the hyperreals are guaranteed to include infinitesimals as well, and so we have at least three levels to the hierarchy: infinities comparable to $H$, finite numbers, and infinitesimals comparable to $1/H$. If you can swallow that, then it's not too much of a leap to add more rungs to the ladder, like extra-small infinitesimals that are comparable to $1/H^2$. If this seems a little crazy, it may comfort you to think of statements about the hyperreals as descriptions of limiting processes involving real numbers. For instance, in the sequence of numbers $1.1^2 = 1.21, 1.01^2 = 1.0201, 1.001^2 = 1.002001, \ldots$, it's clear that the number represented by the digit 1 in the final decimal place is getting smaller faster than the contribution due to the digit 2 in the middle.

One subtle issue here, which I avoided mentioning in the differentiation of the sine function in Example (Page 29), is whether the transfer principle is sufficient to let us define all the functions that appear as familiar keys on a calculator: $x^2$, $\sqrt{x}$, $\sin\ x$, $\cos\ x$, $e^x$, and so on. After all, these functions were originally defined as rules that would take a real number as an input and give a real number as an output. It's not trivially obvious that their definitions can naturally be extended to take a hyperreal number as an input and give back a hyperreal as an output. Essentially the answer is that we can apply the transfer principle to them just as we would to statements about simple arithmetic, but I've discussed this a little more in Is the transfer principle true? (Page 173).

## 2.3 The product rule



When I first learned calculus, it seemed to me that if the derivative of $3t$ was 3, and the derivative of $7t$ was 7, then the derivative of $t$ multiplied by $t$ ought to be just plain old $t$, not $2t$. The reason there's a factor of 2 in the correct answer is that $t^2$ has two reasons to grow as $t$ gets bigger: it grows because the first factor of $t$ is increasing, but also because the second one is. In general, it's possible to find the derivative of the product of two functions any time we know the derivatives of the individual functions.

**The product rule**

If $x$ and $y$ are both functions of $t$, then the derivative of their product is

$$\frac{d(xy)}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

The proof is easy. Changing $t$ by an infinitesimal amount dt changes the product $xy$ by an amount

$$\begin{aligned}&(x+dx)(y+dy)-xy\\&=ydx+xdy+dxdy\end{aligned}$$

and dividing by dt makes this into

$$\frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} + \frac{dxdy}{dt}$$

whose standard part is the result to be proved.

**Example**

Find the derivative of the function $t\ \sin\ t$.

$$\begin{aligned}\frac{d(t\ \sin\ t)}{dt} &= t \cdot \frac{d(\sin\ t)}{dt} + \frac{dt}{dt} \cdot \sin\ t\\&= t\ \cos\ t + \sin\ t\end{aligned}$$

Figure 2.8 gives the geometrical interpretation of the product rule. Imagine that the king, in his castle at the southwest corner of his rectangular kingdom, sends out a line of infantry to expand his territory to the north, and a line of cavalry to take over more land to the east. In a time interval dt, the cavalry, which moves faster, covers a distance $dx$ greater than that covered by the infantry, $dy$. However, the strip of territory conquered by the cavalry, $ydx$, isn't as great as it could have been, because in our example $y$ isn't as big as $x$.

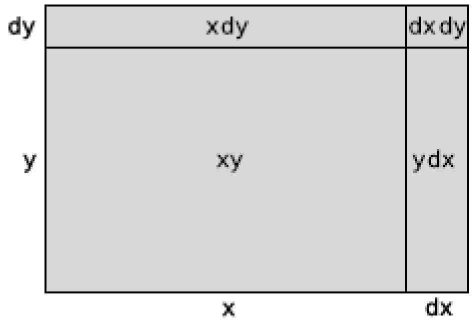


**Fig. 2.8: A geometrical interpretation of the product rule.**

A helpful feature of the Leibniz notation is that one can easily use it to check whether the units of an answer make sense. If we measure distances in meters and time in seconds, then $xy$ has units of square meters (area), and so does the change in the area, $d(xy)$. Dividing by dt gives the number of square meters per second being conquered. On the right-hand side of the product rule, $dx/dt$ has units of meters per second (velocity), and multiplying it by $y$ makes the units square meters per second, which is consistent with the left-hand side. The units of the second term on the right likewise check out. Some beginners might be tempted to guess that the product rule would be $d(xy)/dt = (dx/dt)(dy/dt)$, but the Leibniz notation instantly reveals that this can't be the case, because then the units on the left, $m^2/s$, wouldn't match the ones on the right, $m^2/s^2$.

Because this unit-checking feature is so helpful, there is a special way of writing a second derivative in the Leibniz notation. What Newton called $\ddot{x}$, Leibniz wrote as

$$\frac{d^2x}{dt^2}$$

Although the different placement of the 2's on top and bottom seems strange and inconsistent to many beginners, it actually works out nicely. If $x$ is a distance, measured in meters, and $t$ is a time, in units of seconds, then the second derivative is supposed to have units of acceleration, in units of meters per second per second, also written $(m/s)/s$, or $m/s^2$. (The acceleration of falling objects on Earth is $9.8m/s^2$ in these units.) The Leibniz notation is meant to suggest exactly this: the top of the fraction looks like it has units of meters, because we're not squaring $x$, while the bottom of the fraction looks like it has units of seconds squared, because it looks like we're squaring dt. Therefore the units come out right. It's important to realize, however, that the symbol d isn't a number (not a real one, and not a hyperreal one, either), so we can't really square it; the notation is not to be taken as a literal statement about infinitesimals.

## Example

A tricky use of the product rule is to find the derivative of $\sqrt{t}$. Since $\sqrt{t}$ can be written as $t^{1/2}$, we might suspect that the rule $d(t^k)/dt = kt^{k-1}$ would work, giving a derivative $\frac{1}{2}t^{-1/2} = 1/(2\sqrt{t})$. However, the method from Rates of Change (Page 1) used to prove that rule proved in Derivatives of polynomials (Page 169) only work if $k$ is an integer, so the best we could do would be to confirm our conjecture approximately by graphing or numerical estimation.

Using the product rule, we can write $f(t) = d\sqrt{t}/dt$ for our unknown derivative, and back into the result using the product rule:

$$\begin{aligned}\frac{dt}{dt} &= \frac{d(\sqrt{t}\sqrt{t})}{dt} \\ &= f(t)\sqrt{t} + \sqrt{t}f(t) \\ &= 2f(t)\sqrt{t}\end{aligned}$$

But $dt/dt = 1$, so $f(t) = 1/(2\sqrt{t})$ as claimed.

The trick used in Example (Page 39) can also be used to prove that the power rule $d(x^n)/dx = nx^{n-1}$ applies to cases where $n$ is an integer less than 0, but I'll instead prove this in Example (Page 46) by a technique that doesn't depend on a trick, and also applies to values of $n$ that aren't integers.

## 2.4 The chain rule



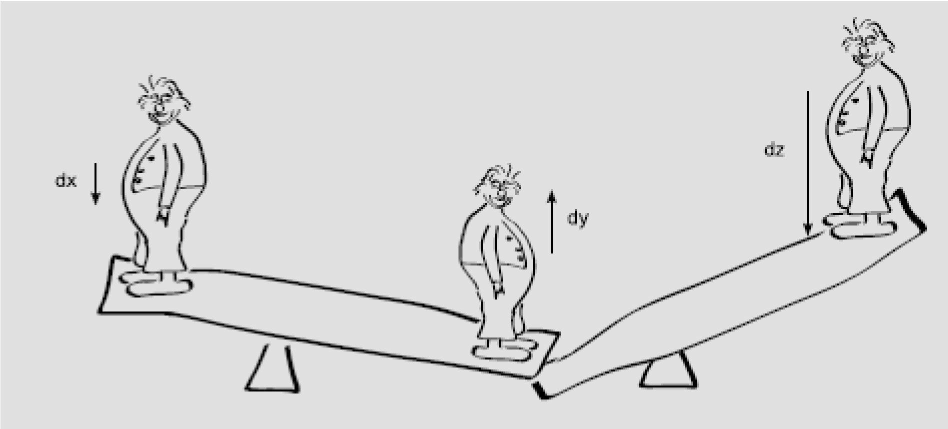


**Fig. 2.9: Three clowns on seesaws demonstrate the chain rule.**

Figure 2.9 shows three clowns on seesaws. If the leftmost clown moves down by a distance $dx$, the middle one will come up by $dy$, but this will also cause the one on the right to move down by $dz$. If we want to predict how much the rightmost clown will move in response to a certain amount of motion by the leftmost one, we have

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

This is called the chain rule. It says that if a change in $x$ causes $y$ to change, and $y$ then causes $z$ to change, then this chain of changes has a cascading effect. Mathematically, there is no big mystery here. We simply cancel $dy$ on the top and bottom. The only minor subtlety is that we would like to be able to be sloppy by using an expression like $dy/dx$ to mean both the quotient of two infinitesimal numbers and a derivative, which is defined as the standard part of this quotient. This sloppiness turns out to be all right, as proved in Proof of the chain rule (Page 178).

## Example

Jane hikes 3 kilometers in an hour, and hiking burns 70 calories per kilometer. At what rate does she burn calories?

We let $x$ be the number of hours she's spent hiking so far, $y$ the distance covered, and $z$ the calories spent. Then

$$\begin{aligned}\frac{dz}{dx} &= \left(\frac{70\text{cal}}{1/\cancel{\text{km}}}\right)\left(\frac{3\ \cancel{\text{km}}}{1\text{hr}}\right)\\ &= 210\text{cal/hr}\end{aligned}$$

## Example

Figure 2.10 shows a piece of farm equipment containing a train of gears with 13, 21, and 42 teeth. If the smallest gear is driven by a motor, relate the rate of rotation of the biggest gear to the rate of rotation of the motor.

Let x, y, and z be the angular positions of the three gears. Then by the chain rule,

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{dy} \cdot \frac{dy}{dx}\\ &= \frac{13}{21} \cdot \frac{21}{42}\\ &= \frac{13}{42}\end{aligned}$$

**Fig. 2.10:**

The chain rule lets us find the derivative of a function that has been built out of one function stuck inside another.

## Example

Find the derivative of the function $z(x) = \sin(x^2)$.

Let $y(x) = x^2$, so that $z(x) = \sin(y(x))$. Then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$
$$= \cos\,(y) \cdot 2x$$
$$= 2x\,\cos(x^2)$$

The way people usually say it is that the chain rule tells you to take the derivative of the outside function, the sine in this case, and then multiply by the derivative of “the inside stuff,” which here is the square. Once you get used to doing it, you don’t need to invent a third, intermediate variable, as we did here with $y$.

## Example

Let’s express the chain rule without the use of the Leibniz notation. Let the function $f$ be defined by $f(x) = g(h(x))$. Then the derivative of $f$ is given by $f'(x) = g'(h(x)) \cdot h'(x)$

## Example

We’ve already proved that the derivative of $t^k$ is $kt^{k-1}$ for $k = -1$ (Example (Page 29)) and for $k = 1, 2, 3, \ldots$ (Derivatives of polynomials (Page 169)). Use these facts to extend the rule to all integer values of $k$.

For $k < 0$, the function $x = t^k$ can be written as $x = (t^{-1})^{-k}$, where $-k$ is positive. Applying the chain rule, we find $dx/dt = (-k)(t^{-1})^{-k-1}(-t^{-2}) = kt^{k-1}$.

## 2.5 Exponentials and logarithms The exponential



The exponential function $e^x$, where $e = 2.71828$ is the base of natural logarithms, comes constantly up in applications as diverse as credit-card interest, the growth of animal populations, and electric circuits. For its derivative we have

$$\frac{de^x}{dx} = \frac{e^{x+dx} - e^x}{dx}$$
$$= \frac{e^x e^{dx} - e^x}{dx}$$
$$= e^x \frac{e^{dx} - 1}{dx}$$

The second factor,$(e^{dx}-1)/dx$, doesn't have $x$ in it, so it must just be a constant. Therefore we know that the derivative of $e^x$ is simply $e^x$, multiplied by some unknown constant,

$$\frac{de^x}{dx} = ce^x$$

A rough check by graphing at, say $x=0$, shows that the slope is close to 1, so $c$ is close to 1. Numerical calculation also shows that, for example, $(e^{0.001}-1)/0.001 = 1.00050016670838$ is very close to 1. But how do we know it's exactly one when dx is really infinitesimal? We can use Inf:

```
: [exp(d)-1]/d
1+0.5d+...
```

(The ... indicates where I've snipped some higher-order terms out of the output.) It seems clear that $c$ is equal to 1 except for negligible terms involving higher powers of $dx$. A rigorous proof is given in Derivative of ex (Page 179).

## Example

The concentration of a foreign substance in the bloodstream generally falls off exponentially with time as $c = c_0 e^{-t/a}$, where $c_0$ is the initial concentration, and a is a constant. For caffeine in adults, a is typically about 7 hours. An example is shown in Figure 2.11. Differentiate the concentration with respect to time, and interpret the result. Check that the units of the result make sense. Using the chain rule,

$$\begin{aligned}\frac{dc}{dt} &= c_0 e^{-t/a} \cdot \left(-\frac{1}{a}\right) \\ &= -\frac{c_0}{a} e^{-t/a}\end{aligned}$$

This can be interpreted as the rate at which caffeine Is being removed from the blood and put into the person’s urine. It’s negative because the concentration is decreasing. According to the original expression for $x$, a substance with a large a will take a long time to reduce its concentration, since $t/a$ won’t be very big unless we have large $t$ on top to compensate for the large $a$ on the bottom. In other words, larger values of a represent substances that the body has a harder time getting rid of efficiently. The derivative has a on the bottom, and the interpretation of this is that for a drug that is hard to eliminate, the rate at which it is removed from the blood is low. It makes sense that a has units of time, because the exponential function has to have a unitless argument, so the units of $t/a$ have to cancel out. The units of the result come from the factor of $c_0/a$, and it makes sense that the units are concentration divided by time, because the result represents the rate at which the concentration is changing.

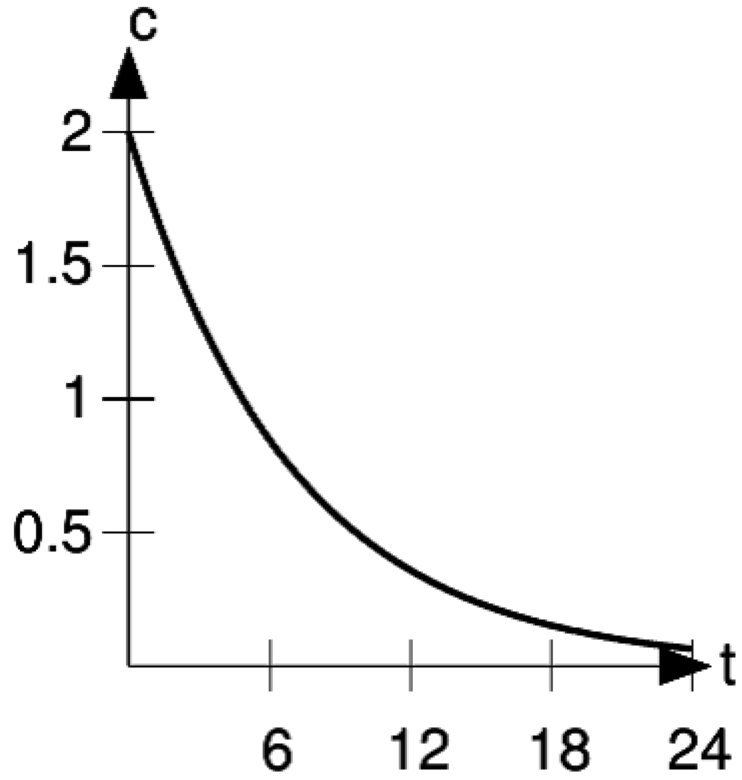


**Fig. 2.11:** A typical graph of the concentration of caffeine in the blood, in units of milligrams per liter, as a function of time, in hours.

## Example

Find the derivative of the function $y = 10^x$ .

In general, one of the tricks to doing calculus is to rewrite functions in forms that you know how to handle. This one can be rewritten as a base-e exponent:

$$y = 10^x$$
$$\text{In} y = \text{In}(10^x)$$
$$\text{In} y = x\text{In}10$$
$$y = e^{x\ \text{In}10}$$

Applying the chain rule, we have the derivative of the exponential, which is just the same exponential, multiplied by the derivative of the inside stuff:

$$\frac{dy}{dx} = e^{x\text{In}10} \cdot \text{In}10$$

In other words, the "c" referred to in the discussion of the derivative of $e^x$ becomes $c = \text{In}10$ in the case of the base-10 exponential.

## 2.5.1 The logarithm



The natural logarithm is the function that undoes the exponential. In a situation like this, we have

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

where on the left we're thinking of $y$ as a function of $x$, and on the right we consider $x$ to be a function of $y$. Applying this to the natural logarithm,

$$\begin{aligned} y &= \text{In} x \\ x &= e^y \\ \frac{dx}{dy} &= e^y \\ \frac{dy}{dx} &= \frac{1}{e^y} \\ &= \frac{1}{x} \\ \frac{\text{dIn } x}{dx} &= \frac{1}{x} \end{aligned}$$

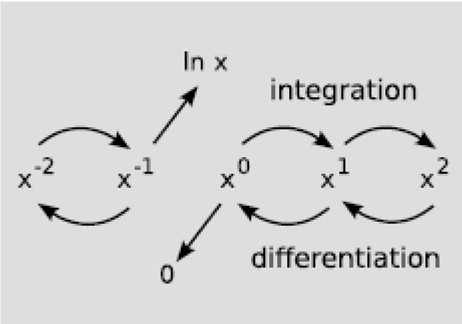


**Fig. 2.12: Differentiation and integration of functions of the form $x^n$. Constants out in front of the functions are not shown, so keep in mind that, for example, the derivative of $x^2$ isn't $x$, it's 2$x$.**

This is noteworthy because it shows that there must be an exception to the rule that the derivative of $x^n$ is $nx^{n-1}$, and the integral of $x^{n-1}$ is $x^n/n$. (In The chain rule (Page 39) I remarked that this rule could be proved using the product rule for negative integer values of $k$, but that I would give a simpler, less tricky, and more general proof later. The proof is Example (Page 46) below.) The integral of $x^{-1}$ is not $x^0/0$, which wouldn't make sense anyway because it involves division by zero. [5] Likewise the derivative of $x^0 = 1$ is $0x^{-1}$, which is zero. Figure 2.12 shows the idea. The functions

5. Speaking casually, one can say that division by zero gives infinity. This is often a good way to think when trying to connect mathematics to reality. However, it doesn't really work that way according to our rigorous treatment of the hyperreals. Consider this statement: \For a nonzero real number a, there is no real number b such that a = 0b." This means that we can't divide a by 0 and get b. Applying the transfer principle to this statement, we see that the same is true for the hyperreals: division by zero is un- defined. However, we can divide a finite number by an infinitesimal, and get an infinite result, which is almost the same thing.

xn form a kind of ladder, with differentiation taking us down one rung, and integration taking us up. However, there are two special cases where differentiation takes us of the ladder entirely.

**Example**

Prove $d(x^n)/dx = nx^{n-1}$ for any real value of $n$, not just an integer.

$$y = x^n$$
$$= e^{\mathrm{nInx}}$$

By the chain rule,

$$\frac{dy}{dx} = e^{\mathrm{nInx}} \cdot \frac{n}{x}$$
$$= x^n \cdot \frac{n}{x}$$
$$= nx^{n-1}$$

(For $n = 0$, the result is zero.)

When I started the discussion of the derivative of the logarithm, I wrote $y = \mathrm{In} x$ right of the bat. That meant I was implicitly assuming $x$ was positive. More generally, the derivative of $\mathrm{In}\ |x|$ equals $1/x$, regardless of the sign (see Problem 2.29 (Page 58) ).

## 2.6 Quotients



So far we've been successful with a divide-and-conquer approach to differentiation: the product rule and the chain rule offer methods of breaking a function down into simpler parts, and finding the derivative of the whole thing based on knowledge of the derivatives of the parts. We know how to find the derivatives of sums, differences, and products, so the obvious next step is to look for a way of handling division. This is straightforward, since we know that the derivative of the function$1/u = u^{-1}$ is $-u^{-2}$.

Let$u$ and $v$ be functions of $x$.

Then by the product rule,

$$\frac{d(v/u)}{dx} = \frac{dv}{dx} \cdot \frac{1}{u} + v \cdot \frac{d(1/u)}{dx}$$

and by the chain rule,

$$\frac{d(v/u)}{dx} = \frac{dv}{dx} \cdot \frac{1}{u} - v \cdot \frac{1}{v^2}\frac{du}{dx}$$

This is so easy to rederive on demand that I suggest not memorizing it.

By the way, notice how the notation becomes a little awkward when we want to write a derivative like $d(v/u)/dx$. When we're differentiating a complicated function, it can be uncomfortable trying to cram the expression into the top of the d.../d... fraction. Therefore it would be more common to write such an expression like this:

$$\frac{d}{dx}\left(\frac{v}{u}\right)$$

This could be considered an abuse of notation, making d look like a number being divided by another number $dx$, when actually $d$ is meaningless on its own. On the other hand, we can consider the symbol $d/dx$ to represent the operation of differentiation with respect to $x$; such an interpretation will seem more natural to those who have been inculcated with the taboo against considering inffnitesimals as numbers in the first place.

Using the new notation, the quotient rule becomes

$$\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{1}{u}\cdot\frac{dv}{dx} - \frac{v}{u^2}\cdot\frac{du}{dx}$$

The interpretation of the minus sign is that if $u$ increases, $v/u$ decreases.

## Example

Differentiate $y = x/(1+3x)$, and check that the result makes sense.

We identify $v$ with $x$ and $u$ with $1+x$.

The result is

$$\begin{aligned}\frac{d}{dx}\left(\frac{v}{u}\right) &= \frac{1}{u}\cdot\frac{dv}{dx} - \frac{v}{u^2}\cdot\frac{du}{dx}\\ &= \frac{1}{1+3x} - \frac{3x}{(1+3x)^2}\end{aligned}$$

One way to check that the result makes sense is to consider extreme values of $x$. For very large values of $x$, the 1 on the bottom of $x/(1+3x)$ becomes negligible compared to the $3x$, and the function $y$ approaches $x/3x = 1/3$ as a limit. Therefore we expect that the derivative $dy/dx$ should approach zero, since the derivative of a constant is zero. It works: plugging in bigger and bigger numbers for $x$ in the expression for the derivative does give smaller and smaller results. (In the second term, the denominator gets bigger faster than the numerator, because it has a square in it.)

Another way to check the result is to verify that the units work out. Suppose arbitrarily that x has units of gallons. (If the 3 on the bottom is unitless, then the 1 would have to represent 1 gallon, since you can't add things that have different units.) The function $y$ is defined by an expression with units of gallons divided by gallons, so $y$ is unitless. Therefore the derivative $dy/dx$ should have units of inverse gallons. Both terms in the expression for the derivative do have those units, so the units of the answer check out.

## 2.7 Differentiation on a computer



In this chapter you've learned a set of rules for evaluating derivatives: derivatives of products, quotients, functions inside other functions, etc. Because these rules exist, it's always possible to find a formula for a function's derivative, given the formula for the original function. Not only that, but there is no real creativity required, so a computer can be programmed to do all the drudgery. For example, you can download a free, opensource program called Yacas from `yacas.sourceforge.net` and install it on a Windows or Linux machine. There is even a version you can run in a web browser without installing any special software: `http://yacas.sourceforge.net/yacasconsole.html`

A typical session with Yacas looks like this:

**Example**

```
D(x) x^2
2*x
D(x) Exp(x^2)
2*x*Exp(x^2)
D(x) Sin(Cos(Sin(x)))
-Cos(x)*Sin(Sin(x))
*Cos(Cos(Sin(x)))
```

Upright type represents your input, and italicized type is the program's output. First I asked it to differentiate $x^2$ with respect to $x$, and it told me the result was $2x$. Then I did the derivative of $e^{x^2}$, which I also could have done fairly easily by hand. (If you're trying this out on a computer as you read along, make sure to capitalize functions like Exp, Sin, and Cos.) Finally I tried an example where I didn't know the answer off the top of my head, and that would have been a little tedious to calculate by hand.

Unfortunately things are a little less rosy in the world of integrals. There are a few rules that can help you do integrals, e.g., that the integral of a sum equals the sum of the integrals, but the rules don't cover all the possible cases. Using Yacas to evaluate the integrals of the same functions, here's what happens. [6]

6. If you're trying these on your own computer, note that the long input line for the function sin cos sin x shouldn't be broken up into two lines as shown in the listing.

## Example

```
Integrate(x) x^2
x^3/3
Integrate(x) Exp(x^2)
Integrate(x)Exp(x^2)
Integrate(x)
Sin(Cos(Sin(x)))
Integrate(x)
Sin(Cos(Sin(x)))
```

The first one works fine, and I can easily verify that the answer is correct, by taking the derivative of $x^3/3$, which is $x^2$. (The answer could have been $x^3/3+7$, or $x^3/3+c$, where $c$ was any constant, but Yacas doesn't bother to tell us that.) The second and third ones don't work, however; Yacas just spits back the input at us without making any progress on it. And it may not be because Yacas isn't smart enough to figure out these integrals. The function $e^{x^2}$ can't be integrated at all in terms of a formula containing ordinary operations and functions such as addition, multiplication, exponentiation, trig functions, exponentials, and so on.

That's not to say that a program like this is useless. For example, here's an integral that I wouldn't have known how to do, but that Yacas handles easily:

## Example

```
Integrate(x) Sin(Ln(x))
(x*Sin(Ln(x)))/2
-(x*Cos(Ln(x)))/2
```

This one is easy to check by differentiating, but I could have been marooned on a desert island for a decade before I could have figured it out in the first place. There are various rules, then, for integration, but they don't cover all possible cases as the rules for differentiation do, and sometimes it isn't obvious which rule to apply. Yacas's ability to integrate sin ln $x$ shows that it had a rule in its bag of tricks that I don't know, or didn't remember, or didn't realize applied to this integral.

Back in the 17th century, when Newton and Leibniz invented calculus, there were no computers, so it was a big deal to be able to find a simple formula for your result. Nowadays, however, it may not be such a big deal. Suppose I want to find the derivative of sin cos $\sin x$, evaluated at $x = 1$. I can do something like this on a calculator:

### Example

```
sin cos sin 1 =
0.61813407
sin cos sin 1.0001 =
0.61810240
(0.61810240-0.61813407)
/.0001 =
-0.3167
```

I have the right answer, with plenty of precision for most realistic applications, although I might have never guessed that the mysterious number 0.3167 was actually (cos 1)(sin sin 1)(cos cos sin 1). This could get a little tedious if I wanted to graph the function, for instance, but then I could just use a computer spreadsheet, or write a little computer program. In this chapter, I'm going to show you how to do derivatives and integrals using simple computer programs, using Yacas. The following little Yacas program does the same thing as the set of calculator operations shown above:

### Example

```
1 f(x):=Sin(Cos(Sin(x)))
2 x:=1
3 dx:=.0001
4 N( (f(x+dx)-f(x))/dx )
-0.3166671628
```

(I've omitted all of Yacas's output except for the _nal result.) Line 1 de_nes the function we want to di_erentiate. Lines 2 and 3 give values to the variables x and dx. Line 4 computes the derivative; the N( ) surrounding the whole thing is our way of telling Yacas that we want an approximate numerical result, rather than an exact symbolic one.

An interesting thing to try now is to make dx smaller and smaller, and see if we get better and better accuracy in our approximation to the derivative.

# Example

```
5 g(x,dx):=
N( (f(x+dx)-f(x))/dx )
6 g(x,.1)
-0.3022356406
7 g(x,.0001)
-0.3166671628
8 g(x,.0000001)
-0.3160458019
9 g(x,.00000000000000001)
0
```

Line 5 defines the derivative function. It needs to know both $x$ and $dx$. Line 6 computes the derivative using dx = 0.1, which we expect to be a lousy approximation, since $dx$ is really supposed to be infinitesimal, and 0.1 isn't even that small. Line 7 does it with the same value of $dx$ we used earlier. The two results agree exactly in the first decimal place, and approximately in the second, so we can be pretty sure that the derivative is -0.32 to two figures of precision. Line 8 ups the ante, and produces a result that looks accurate to at least 3 decimal places. Line 9 attempts to produce fantastic precision by using an extremely small value of dx. Oops - the result isn't better, it's worse! What's happened here is that Yacas computed $f(x)$ and $f(x+dx)$, but they were the same to within the precision it was using, so $f(x+dx)-f(x)$ rounded off to zero. [7]

Example (Page 51) demonstrates the concept of how a derivative can be defined in terms of a limit:

$$\frac{dy}{dx} = \lim_{\Delta \to 0} \frac{\Delta y}{\Delta x}$$

The idea of the limit is that we can theoretically make $\Delta y/\Delta x$ approach as close as we like to $dy/dx$, provided we make $\Delta x$ sufficiently small. In reality, of course, we eventually run into the limits of our ability to do the computation, as in the bogus result generated on line 9 of the example.

## 2.8 Problems

### 2.8.1 Problem 2.1



Carry out a calculation like the one in Example (Page 27) to show that the derivative of $t^4$ equals $4t^3$.

7. Yacas can do arithmetic to any precision you like, although you may run into practical limits due to the amount of memory your computer has and the speed of its CPU. For fun, try N(Pi,1000), which tells Yacas to compute π numerically to 1000 decimal places.

Solutions for chapter 2 (Page 196)

### 2.8.2 Problem 2.2



Example (Page 30) gave a tricky argument to show that the derivative of $\cos\ t$ is $\sin\ t$. Prove the same result using the method of Example (Page 29) instead.

Solutions for chapter 2 (Page 196)

### 2.8.3 Problem 2.3



Suppose $H$ is a big number. Experiment on a calculator to figure out whether $\sqrt{H+1}-\sqrt{H-1}$ comes out big, normal, or tiny. Try making $H$ bigger and bigger, and see if you observe a trend. Based on these numerical examples, form a conjecture about what happens to this expression when $H$ is infinite.

Solutions for chapter 2 (Page 196)

### 2.8.4 Problem 2.4



Suppose $dx$ is a small but finite number. Experiment on a calculator to figure out how $\sqrt{dx}$ compares in size to $dx$. Try making $dx$ smaller and smaller, and see if you observe a trend. Based on these numerical examples, form a conjecture about what happens to this expression when $dx$ is infinitesimal.

Solutions for chapter 2 (Page 196)

### 2.8.5 Problem 2.5



To which of the following statements can the transfer principle be applied? If you think it can't be applied to a certain statement, try to prove that the statement is false for the hyperreals, e.g., by giving a counterexample.

a. For any real numbers $x$ and $y$, $x+y=y+x$.
b. The sine of any real number is between -1 and 1.
c. For any real number $x$, there exists another real number $y$ that is greater than $x$.
d. For any real numbers $x \neq y$, there exists another real number $z$ such that $x<z<y$.
e. For any real numbers $x \neq y$, there exists a rational number $z$ such that $x<z<y$. (A rational number is one that can be expressed as an integer divided by another integer.)

f. For any real numbers $x$, $y$, and $z$, $(x+y)+z=x+(y+z)$.
g. For any real numbers $x$ and $y$, either $x<y$ or $x=y$ or $x>y$
h. For any real number $x$, $x+1\neq x$.

Solutions for chapter 2 (Page 196)

## 2.8.6 Problem 2.6



If we want to pump air or water through a pipe, common sense tells us that it will be easier to move a larger quantity more quickly through a fatter pipe. Quantitatively, we can define the resistance, $R$, which is the ratio of the pressure difference produced by the pump to the rate of flow. A fatter pipe will have a lower resistance. Two pipes can be used in parallel, for instance when you turn on the water both in the kitchen and in the bathroom, and in this situation, the two pipes let more water flow than either would have let flow by itself, which tells us that they act like a single pipe with some lower resistance. The equation for their combined resistance is $R=1/(1/R_1+1/R_2)$. Analyze the case where one resistance is finite, and the other infinite, and give a physical interpretation. Likewise, discuss the case where one is finite, but the other is infinitesimal.

Solutions for chapter 2 (Page 196)

## 2.8.7 Problem 2.7



Naively, we would imagine that if a spaceship traveling at $u=3/4$ of the speed of light was to shoot a missile in the forward direction at $v=3/4$ of the speed of light (relative to the ship), then the missile would be traveling at $u+v=3/2$ of the speed of light. However, Einstein's theory of relativity tells us that this is too good to be true, because nothing can go faster than light. In fact, the relativistic equation for combining velocities in this way is not $u+v$, but rather $(u+v)/(1+uv)$. In ordinary, everyday life, we never travel at speeds anywhere near the speed of light. Show that the nonrelativistic result is recovered in the case where both $u$ and $v$ are infinitesimal.

Solutions for chapter 2 (Page 196)

## 2.8.8 Problem 2.8



Differentiate $(2x+3)^{100}$ with respect to $x$.

Solutions for chapter 2 (Page 196)

### 2.8.9 Problem 2.9



1. Differentiate $(x+1)^{100}(x+2)^{200}$ with respect to $x$.

Solutions for chapter 2 (Page 196)

### 2.8.10 Problem 2.10



Differentiate the following with respect to $x$: $e^{7x}$, $e^{e^x}$ . (In the latter expression, as in all exponentials nested inside exponentials, the evaluation proceeds from the top down, i.e., $e^{(e^x)}$, not $(e^e)^x$) .

Solutions for chapter 2 (Page 196)

### 2.8.11 Problem 2.11



Differentiate $a \sin(bx+c)$ with respect to $x$.

Solutions for chapter 2 (Page 196)

### 2.8.12 Problem 2.12



Let $x = t^{p/q}$, where $p$ and $q$ are positive integers. By a technique similar to the one in Example (Page 42), prove that the differentiation rule for $t^k$ holds when $k = p/q.qwe$ .

Solutions for chapter 2 (Page 196)

### 2.8.13 Problem 2.13



Find a function whose derivative with respect to $x$ equals $a \sin(bx+c)$. That is, find an integral of the given function.

Solutions for chapter 2 (Page 196)

## 2.8.14 Problem 2.14



Use the chain rule to differentiate $((x^2)^2)^2$, and show that you get the same result you would have obtained by differentiating $x^8$.

Solutions for chapter 2 (Page 196) [8]

## 2.8.15 Problem 2.15



The range of a gun, when elevated to an angle$\theta$, is given by

$$R = \frac{2v^2}{g}\sin\theta\cos\theta$$

Find the angle that will produce the maximum range.

Solutions for chapter 2 (Page 196)

## 2.8.16 Problem 2.16



Differentiate $\sin\cos\tan x$ with respect to $x$.

## 2.8.17 Problem 2.17



The hyperbolic cosine function is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Find any minima and maxima of this function.

Solutions for chapter 2 (Page 196)

## 2.8.18 Problem 2.18



Show that the function $\sin(\sin(\sin x))$ has maxima and minima at all the same places where $\sin x$ does, and at no other places.

Solutions for chapter 2 (Page 196)

8. [M. Livshits]

## 2.8.19 Problem 2.19



Let $f(x) = |x| + x$ and $g(x) = x\,|x| + x$. Find the derivatives of these functions at $x = 0$ in terms of (a) slopes of tangent lines and (b) infinitesimals.

Solutions for chapter 2 (Page 196)

## 2.8.20 Problem 2.20



In free fall, the acceleration will not be exactly constant, due to air resistance. For example, a skydiver does not speed up indefinitely until opening her chute, but rather approaches a certain maximum velocity at which the upward force of air resistance cancels out the force of gravity. The expression for the distance dropped by of a free-falling object, with air resistance, is [9]

$$d = A \text{ In} \left[\cosh\left(t\sqrt{\frac{g}{A}}\right)\right]$$

where $g$ is the acceleration the object would have without air resistance, the function cosh has been defined in Problem 2.17 (Page 55), and $A$ is a constant that depends on the size, shape, and mass of the object, and the density of the air. (For a sphere of mass $m$ and diameter $d$ dropping in air, $A = 4.11m/d^2$. Cf. Problem 2.10 (Page 54))

1. (a) Differentiate this expression to find the velocity. Hint: In order to simplify the writing, start by defining some other symbol to stand for the constant $\sqrt{g/A}$.
2. (b) Show that your answer can be reexpressed in terms of the function tanh defined by tanh $x = (e^x - e^{-x})/(e^x + e^{-x})$.
3. (c) Show that your result for the velocity approaches a constant for large values of $t$.
4. (d) Check that your answers to parts b and c have units of velocity.

Solutions for chapter 2 (Page 196)

## 2.8.21 Problem 2.21



Differentiate $\tan\,\theta$ with respect to $\theta$.

Solutions for chapter 2 (Page 196)

9. Jan Benacka and Igor Stubna, The Physics Teacher, 43 (2005) 432.

## 2.8.22 Problem 2.22



Differentiate $\sqrt[3]{x}$ with respect to $x$.

Solutions for chapter 2 (Page 196)

## 2.8.23 Problem 2.23



Differentiate the following with respect to $x$:

a. $y = \sqrt{x^2+1}$
b. $y = \sqrt{x^2+a^2}$
c. $y = 1/\sqrt{a+x}$
d. $y = a/\sqrt{a-x^2}$.

Solutions for chapter 2 (Page 196)

## 2.8.24 Problem 2.24



Differentiate $\text{In}(2t+1)$ with respect to $t$.

Solutions for chapter 2 (Page 196)

## 2.8.25 Problem 2.25



If you know the derivative of $\sin\ x$, it's not necessary to use the product rule in order to differentiate $3\sin\ x$, but show that using the product rule gives the right result anyway.

Solutions for chapter 2 (Page 196)

## 2.8.26 Problem 2.26



The $\Gamma$ function (capital Greek letter gamma) is a continuous mathematical function that has the property $\Gamma(n) = 1 \cdot 2 \cdot \ldots \cdot (n-1)$ for n an integer. $\Gamma(x)$ is also well defined for values of $x$ that are not integers, e.g., $\Gamma(1/2)$ happens to be $\sqrt{\pi}$. Use computer software that is capable of evaluating the $\Gamma$ function to determine numerically the derivative of $\Gamma(x)$ with respect to $x$, at $x = 2$. (In Yacas, the function is called Gamma.)

Solutions for chapter 2 (Page 196)

## 2.8.27 Problem 2.27



For a cylinder of fixed surface area, what proportion of length to radius will give the maximum volume?

Solutions for chapter 2 (Page 196)

## 2.8.28 Problem 2.28



This problem is a variation on Problem 2.11 (Page 54). Einstein found that the equation $K = (1/2)mv^2$ for kinetic energy was only a good approximation for speeds much less than the speed of light, $c$. At speeds comparable to the speed of light, the correct equation is

$$K = \frac{\frac{1}{2}mv^2}{\sqrt{1 - v^2/c^2}}$$

a. As in the earlier, simpler problem, find the power $dK/dt$ for an object accelerating at a steady rate, with $v = at$.
b. Check that your answer has the right units.
c. Verify that the power required becomes infinite in the limit as $v$ approaches $c$, the speed of light. This means that no material object can go as fast as the speed of light.

Solutions for chapter 2 (Page 196)

## 2.8.29 Problem 2.29



Prove, as claimed in The logarithm (Page 45), that the derivative of $\ln |x|$ equals $1/x$, for both positive and negative $x$.

Solutions for chapter 2 (Page 196)

## 2.8.30 Problem 2.30



On even function is one with the property $f(-x) = f(x)$. For example, $\cos x$ is an even function, and $x^n$ is an even function if $n$ is even. An odd function has $f(-x) = -f(x)$. Prove that the derivative of an even function is odd.

Solutions for chapter 2 (Page 196)

### 2.8.31 Problem 2.31



Suppose we have a list of numbers $x_1, \ldots x_n$, and we wish to find some number $q$ that is as close as possible to as many of the $x_i$ as possible. To make this a mathematically precise goal, we need to define some numerical measure of this closeness. Suppose we let $h = (x_1 - q)^2 + \ldots + (x_n - q)^2$, which can also be notated using $\sum$, uppercase Greek sigma, as $h = \sum_{i=1}^{n} (x_i - q)^2$. Then minimizing $h$ can be used as a definition of optimal closeness. (Why would we not want to use $h = \sum_{i=1}^{n} (x_i - q)$?) Prove that the value of $q$ that minimizes $h$ is the average of the $x_i$.

### 2.8.32 Problem 2.32



Use a trick similar to the one used in Example (Page 39) to prove that the power rule $d(x^k)/dx = kx^{k-1}$ applies to cases where k is an integer less than 0.

Solutions for chapter 2 (Page 196)

### 2.8.33 Problem 2.33



The plane of Euclidean geometry is today often described as the set of all coordinate pairs $(x, y)$, where $x$ and $y$ are real. We could instead imagine the plane F that is defined in the same way, but with $x$ and $y$ taken from the set of hyperreal numbers. As a third alternative, there is the plane G in which the finite hyperreals are used. In E, Euclid's parallel postulate holds: given a line and a point not on the line, there exists exactly one line passing through the point that does not intersect the line. Does the parallel postulate hold in F? In G? Is it valid to associate only E with the plane described by Euclid's axioms? .

Solutions for chapter 2 (Page 196)

### 2.8.34 Problem 2.34



Discuss the following statement: *The repeating decimal 0.999... is infinitesimally less than one.*

Solutions for chapter 2 (Page 196)

## 2.8.35 Problem 2.35



Example (Page 42) expressed the chain rule without the Leibniz notation, writing a function $f$ defined by $f(x)=g(h(x))$. Suppose that you're trying to remember the rule, and two of the possibilities that come to mind are $f'(x)=g'(h(x))$ and $f'(x)=g'(h(x))h(x)$. Show that neither of these can possibly be right, by considering the case where $x$ has units. You may _nd it helpful to convert both expressions back into the Leibniz notation.

Solutions for chapter 2 (Page 196)

## 2.8.36 Problem 2.36



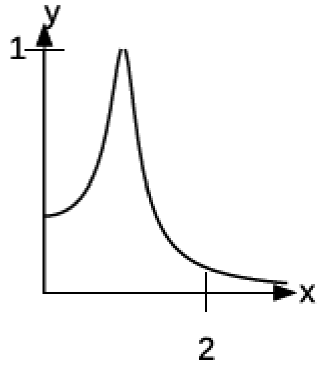


**Fig. 2.13: The function of Problem 2.36, with** $a=3, b=1, \text{and } f_o=1$

When you tune in a radio station using an old-fashioned rotating dial you don't have to be exactly tuned in to the right frequency in order to get the station. If you did, the tuning would be infinitely sensitive, and you'd never be able to receive any signal at all! Instead, the tuning has a certain amount of "slop" intentionally designed into it. The strength of the received signal $s$ can be expressed in terms of the dial's setting $f$ by a function of the form

$$s=\frac{1}{\sqrt{a(f^2-f_o^2)^2+bf^2}}$$

where $a$, $b$, and $f_o$ are constants. This functional form is in fact very general, and is encountered in many other physical contexts. The graph below shows the resulting bell-shaped curve. Find the frequency $f$ at which the maximum response occurs, and show that if $b$ is small, the maximum occurs close to, but not exactly at, $f_o$.

Solutions for chapter 2 (Page 196)

## 2.8.37 Problem 2.37



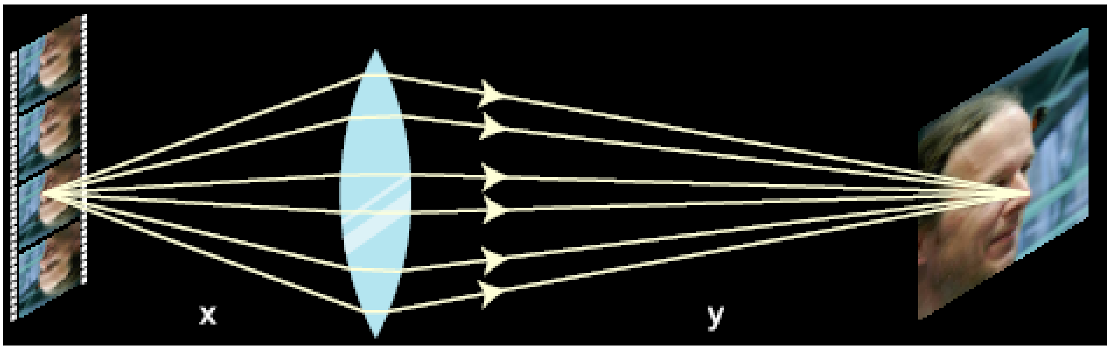


**Fig. 2.14:** A set of light rays is emitted from the tip of the glamorous movie star's nose on the film, and reunited to form a spot on the screen which is the image of the same point on his nose. The distances have been distorted for clarity. The distance y represents the entire length of the theater from front to back.

In a movie theater, the image on the screen is formed by a lens in the projector, and originates from one of the frames on the strip of celluloid film (or, in the newer digital projection systems, from a liquid crystal chip). Let the distance from the film to the lens be $x$, and let the distance from the lens to the screen be $y$. The projectionist needs to adjust $x$ so that it is properly matched with $y$, or else the image will be out of focus. There is therefore a fixed relationship between $x$ and $y$, and this relationship is of the form

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{f}$$

where $f$ is a property of the lens, called its focal length. A stronger lens has a shorter focal length. Since the theater is large, and the projector is relatively small, $x$ is much less than $y$. We can see from the equation that if $y$ is sufficiently large, the left-hand side of the equation is dominated by the $1/x$ term, and we have $x \approx f$. Since the $1/y$ term doesn't completely vanish, we must have $x$ slightly greater than $f$, so that the $1/x$ term is slightly less than $1/f$. Let $x = f + dx$, and approximate $dx$ as being infinitesimally small. Find a simple expression for $y$ in terms of $f$ and $dx$.

Solutions for chapter 2 (Page 196)

## 2.8.38 Problem 2.38



Why might the expression $1_\infty$ be considered an indeterminate form? .

Solutions for chapter 2 (Page 196)

# Chapter 3 Limits and continuity

## 3.1 Continuity



Intuitively, a continuous function is one whose graph has no sudden jumps in it; the graph is all a single connected piece. Such a function can be drawn without picking the pen up off of the paper. Formally, a function $f(x)$ is defined to be continuous if for any real $x$ and any infinitesimal $dx$, $f(x+dx)-f(x)$ is infinitesimal.

**Example**

Let the function $f$ be defined by $f(x)=0$ for $x \leq 0$, and $f(x)=1$ for $x>0$. Then f (x ) is discontinuous, since for dx > 0, f (0 + dx ) − f (0) = 1, which isn't infinitesimal.

**Fig. 3.1: Caption** The black dot indicates that the endpoint of the lower ray is part of the ray, while the white one shows the contrary for the ray on the top

If a function is discontinuous at a given point, then it is not differentiable at that point. On the other hand, the example $y=|x|$ shows that a function can be continuous without being differentiable.

In most cases, there is no need to invoke the definition explicitly in order to check whether a function is continuous. Most of the functions we work with are de- fined by putting together simpler functions as building blocks. For example, let's say we're already convinced that the functions defined by $g(x)=3x$ and $h(x)=\sin x$ are both continuous. Then if we encounter the function $f(x)=\sin(3x)$, we can tell that it's continuous because its definition corresponds to $f(x)=h(g(x))$. The functions $g$ and $h$ have been set up like a bucket brigade, so that $g$ takes the input, calculates the output, and then hands it off to $h$ for the final step of the calculation. This method of combining functions is called *composition*. The composition of two continuous functions is also continuous. Just watch out for division. The function $f(x)=1/x$ is continuous everywhere except at $x=0$, so for example $1/\sin(x)$ is continuous everywhere except at multiples of $\pi$, where the sine has zeroes.

## 3.1.1 The intermediate value theorem



Another way of thinking about continuous functions is given by the *intermediate value theorem*. Intuitively, it says that if you are moving continuously along a road, and you get from point A to point B, then you must also visit every other point along the road; only by teleporting (by moving discontinuously) could you avoid doing so. More formally, the theorem states that if $y$ is a continuous real-valued function on the real interval from a to $b$, and if $y$ takes on values $y_1$ and $y_2$ at certain points within this interval, then for any $y_3$ between $y_1$ and $y_2$, there is some real $x$ in the interval for which $y(x) = y_3$.

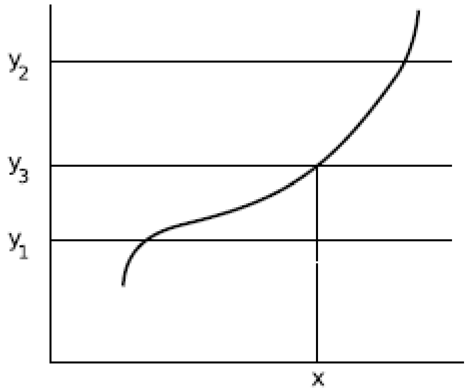


**Fig. 3.2:** The intermediate value theorem states that if the function is continuous, it must pass through

$$y_3$$

The intermediate value theorem seems so intuitively appealing that if we want to set out to prove it, we may feel as though we're being asked to prove a proposition such as, "a number greater than 10 exists." If a friend wanted to bet you a six-pack that you couldn't prove this with complete mathematical rigor, you would have to get your friend to spell out very explicitly what she thought were the facts about integers that you were allowed to start with as initial assumptions. Are you allowed to assume that 1 exists? Will she grant you that if a number n exists, so does $n + 1$? The intermediate value theorem is similar. It's stated as a theorem about certain types of functions, but its truth isn't so much a matter of the properties of functions as the properties of the underlying number system. For the reader with a interest in pure mathematics, I've discussed this in more detail in The intermediate value theorem (Page 63) and given an abbreviated proof. (Most introductory calculus texts do not prove it at all.)

## Example

Show that there is a solution to the equation $10^x + x = 1000$.

We expect there to be a solution near $x = 3$, where the function $f(x) = 10^x + x = 1003$ is just a little too big. On the other hand, $f(2) = 102$ is much too small. Since $f$ has values above and below 1000 on the interval from 2 to 3, and $f$ is continuous, the intermediate value theorem proves that a solution exists between 2 and 3. If we wanted to find a better numerical approximation to the solution, we could do it using Newton's method, which is introduced in Newton's method (Page 101).

## Example

Show that there is at least one solution to the equation $\cos x = x$, and give bounds on its location.

This is a transcendental equation, and no amount of fiddling with algebra and trig identities will ever give a closed-form solution, i.e., one that can be written down with a finite number of arithmetic operations to give an exact result. However, we can easily prove that at least one solution exists, by applying the intermediate value theorem to the function $x - \cos x$. The cosine function is bounded between −1 and 1, so this function must be negative for $x < -1$ and positive for $x > 1$. By the intermediate value theorem, there must be a solution in the interval $-1 \leq x \leq 1$. The graph, c, verifies this, and shows that there is only one solution.

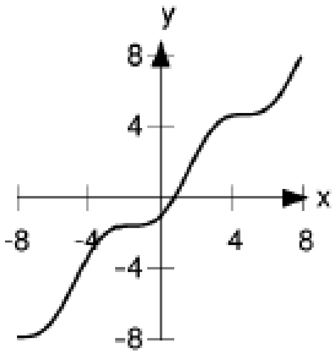


**Fig. 3.3:**

$x - \cos x$ constructed in Example (Page 64).

## Example

Prove that every odd-order polynomial $P$ with real coefficients has at least one real root $x$ , i.e., a point at which $P(x) = 0$.

Example (Page 64) might have given the impression that there was nothing to be learned from the intermediate value theorem that couldn't be deter- mined by graphing, but this example clearly can't be solved by graphing, because we're trying to prove a general result for all polynomials.

To see that the restriction to odd orders is necessary, consider the polynomial $x^2 + 1$, which has no real roots because $x^2 > 0$ for any real number $x$.

To fix our minds on a concrete example for the odd case, consider the polynomial $P(x) = x^3 - x + 17$. For large values of $x$, the linear and constant terms will be negligible compared to the $x^3$ term, and since $x^3$ is positive for large values of $x$ and negative for large negative ones, it follows that $P$ is sometimes positive and sometimes negative.

Making this argument more general and rigorous, suppose we had a polynomial of odd order $n$ that always had the same sign for real $x$. Then by the transfer principle the same would hold for any hyperreal value of $x$. Now if $x$ is infinite then the lower-order terms are infinitesimal compared to the $x^n$ term, and the sign of the result is determined entirely by the $x^n$ term, but $x^n$ and $(-x)^n$ have opposite signs, and therefore $P(x)$ and $P(-x)$ have opposite signs. This is a contradiction, so we have disproved the assumption that $P$ always had the same sign for real $x$. Since $P$ is sometimes negative and sometimes positive, we conclude by the intermediate value theorem that it is zero somewhere.

## Example

Show that the equation $x = \sin 1/x$ has infinitely many solutions.

This is another example that can't be solved by graphing; there is clearly no way to prove, just by looking at a graph like d, that it crosses the x axis *infinitely* many times. The graph does, however, help us to gain intuition for what's going on. As $x$ gets smaller and smaller, $1/x$ blows up, and $\sin 1/x$ oscillates more and more rapidly. The function $f$ is undefined at 0, but it's continuous everywhere else, so we can apply the intermediate value theorem to any interval that doesn't include 0.

We want to prove that for any positive $u$, there exists an $x$ with $0 < x < u$ for which $f(x)$ has either desired sign. Suppose that this fails for some real $u$. Then by the transfer principle the nonexistence of any real $x$ with the desired property also implies the nonexistence of any such hyperreal $x$. But for an infinitesimal $x$ the sign of $f$ is determined entirely by the sine term, since the sine term is finite and the linear term infinitesimal. Clearly $\sin 1/x$ can't have a single sign for all values of $x$ less than $u$, so this is a contradiction, and the proposition succeeds for any u. It follows from the intermediate value theorem that there are infinitely many solutions to the equation.

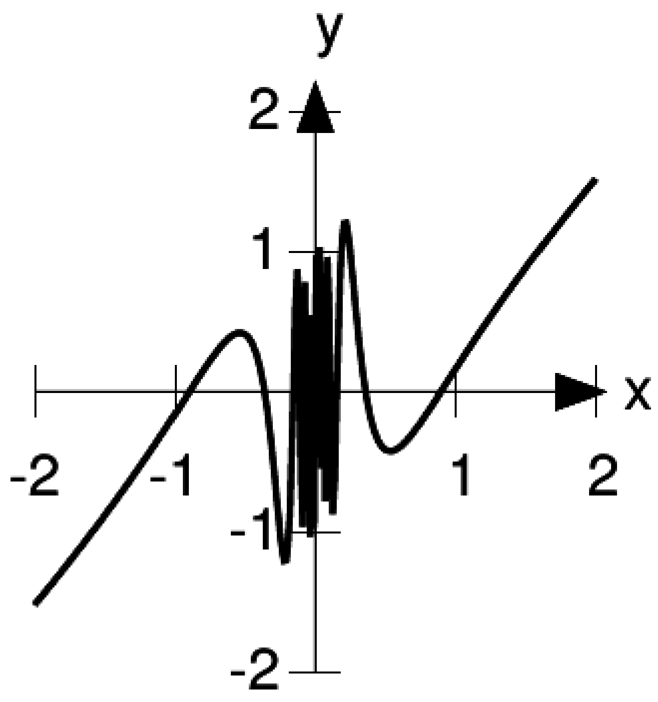


**Fig. 3.4: The function**

$x - \sin 1/x$

## 3.1.2 The extreme value theorem



In chapter 1, we saw that locating maxima and minima of functions may in general be fairly difficult, because there are so many different ways in which a function can attain an extremum: e.g., at an endpoint, at a place where its derivative is zero, or at a non-differentiable kink. The following theorem allows us to make a very general statement about all these possible cases, assuming only continuity.

The *extreme value theorem* states that if $f$ is a continuous real-valued function on the real-number interval defined by $a \leq x \leq b$, then $f$ has maximum and minimum values on that interval, which are attained at specific points in the interval.

Let's first see why the assumptions are necessary. If we weren't confined to a finite interval, then $y = x$ would be a counterexample, because it's continuous and doesn't have any maximum or minimum value. If we didn't assume continuity, then we could have a function defined as $y = x$ for $x < 1$, and $y = 0$ for $x \geq 1$; this function never gets bigger than 1, but it never attains a value of 1 for any specific value of $x$.

The extreme value theorem is proved, in a somewhat more general form,in Proof of the extreme value theorem (Page 185).

### Example

Find the maximum value of the polynomial $P(x) = x^3 + x^2 + x + 1$ for $-5 \leq x \leq 5$.

Polynomials are continuous, so the extreme value theorem guarantees that such a maximum exists. Suppose we try to find it by looking for a place where the derivative is zero. The derivative is $3x^2 + 2x + 1$, and setting it equal to zero gives a quadratic equation, but application of the quadratic formula shows that it has no real solutions. It appears that the function doesn't have a maximum anywhere (even outside the interval of interest) that looks like a smooth peak. Since it doesn't have kinks or discontinuities, there is only one other type of maxi- mum it could have, which is a maxi- mum at one of its endpoints. Plugging in the limits, we find $P(-5) = -104$ and $P(5) = 156$, so we conclude that the maximum value on this interval is 156.

## 3.2 Limits



Historically, the calculus of infinitesimals as created by Newton and Leibniz was reinterpreted in the nineteenth century by Cauchy, Bolzano, and Weierstrass in terms of limits. All mathematicians learned both languages, and switched back and forth between them effortlessly, like the lady I overheard in a Southern California supermarket telling her mother, “Let’s get that one, con los nuts.” Those who had been trained in infinitesimals might hear a statement using the language of limits, but translate it mentally into infinitesimals; to them, every statement about limits was really a statement about infinitesimals. To their younger colleagues, trained using limits, every statement about infinitesimals was really to be under- stood as shorthand for a limiting process. When Robinson laid the rigorous foundations for the hyper- real number system in the 1960’s, a common objection was that it was really nothing new, because every statement about infinitesimals was really just a different way of expressing a corresponding statement about limits; of course the same could have been said about Weierstrass’s work of the preceding century! In reality, all practitioners of calculus had realized all along that different approaches worked better for different problems; Problem 4.13 (Page 100) is an example of a result that is much easier to prove with infinitesimals than with limits.

The Weierstrass definition of a limit is this:

## Definition of the limit

We say that $l$ is the limit of the function $f(x)$ as $x$ approaches $a$, written

$$\lim_{x\to a} f(x) = l$$

if the following is true: for any real number $\epsilon$, there exists another real number $\delta$ such that for all $x$ in the interval $a-\delta \le x \le a+\delta$, the value of $f$ lies within the range from $l-\epsilon$ to $l+\epsilon$.

Intuitively, the idea is that if I want you to make $f(x)$ close to $l$ ,just have to tell you how close, and you can tell me that it will be that close as long as x is within a certain distance of a.

In terms of infinitesimals, we have:

## Definition of the limit

We say that $l$ is the limit of the function $f(x)$ as $x$approaches $a$, written

$$\lim_{x\to a} f(x) = l$$

if the following is true: for any infinitesimal number dx, the value of $f(a+dx)$ is finite, and the standard part of $f(a+dx)$ equals $l$.

The two definitions are equivalent. As remarked previously, the derivative $dx/dt$ can be defined as the limit $\lim \Delta_{t\to 0}(\Delta x/\Delta t)$, and if we use the Weierstrass definition of the limit, this means that the derivative can be defined entirely in terms of the real number sys- tem, without the user of hyperreal numbers.

Sometimes a limit can be evaluated simply by plugging in numbers:

## Example

Evaluate

$$\lim_{x \to 0} \frac{1}{1+x}$$

Plugging in $x = 0$, we find that the limit is 1.

In some examples, plugging in fails if we try to do it directly, but can be made to work if we massage the expression into a different form:

## Example

Evaluate

$$\lim_{x \to 0} \frac{\frac{2}{x} + 7}{\frac{1}{x} + 8686}$$

Plugging in $x = 0$ fails because division by zero is undefined.

Intuitively, however, we expect that the limit will be well defined, and will equal 2, because for very small values of $x$ , the numerator is dominated by the $2/x$ term, and the denominator by the $1/x$ term, so the 7 and 8686 terms will matter less and less as $x$ gets smaller and smaller.

To demonstrate this more rigorously, a trick that works is to multiply both the top and the bottom by $x$ , giving

$$\frac{2+7x}{1+8686x}$$

which equals 2 when we plug in $x = 0$, so we find that the limit is zero.

This example is a little subtle, because when $x$ equals zero, the function is not defined, and moreover it would not be valid to multiply both the top and the bottom by $x$. In general, it's not valid algebra to multiply both the top and the bottom of a fraction by 0, because the result is 0/0, which is undefined. But we didn't actually multiply both the top and the bottom by zero, because we never let $x$ equal zero. Both the Weierstrass definition and the definition in terms of infinitesimals only refer to the properties of the function in a region very close to the limiting point, not at the limiting point itself.

This is an example in which the function was not well defined at a certain point, and yet the limit of the function was well defined as we approached that point. In a case like this, where there is only one point missing from the domain of the function, it is natural to extend the definition of the function by filling in the "gap tooth." Example

(Page 71) below shows that this kind of filling- in procedure is not always possible.

# Example

Investigate the limiting behavior of $1/x^2$ as $x$ approaches 0, and 1.

At $x = 1$, plugging in works, and we find that the limit is 1.

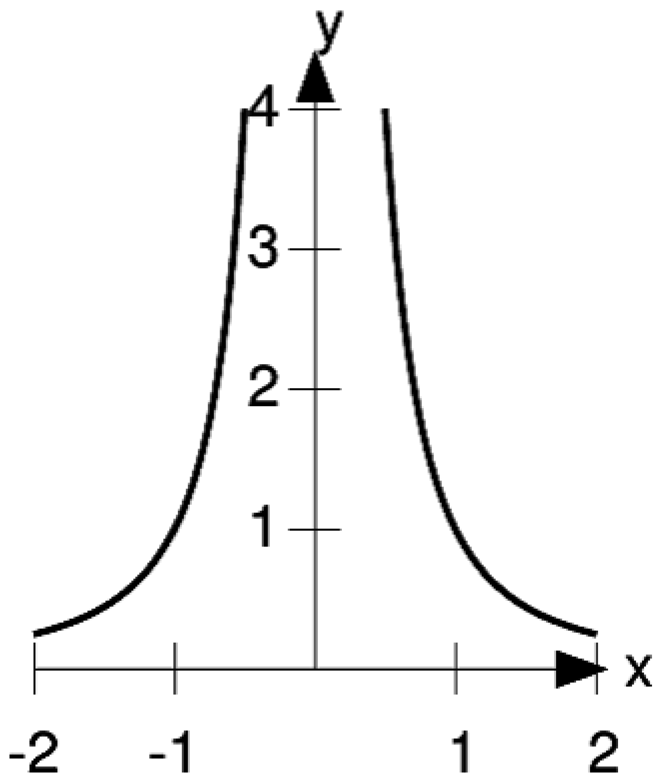


**Fig. 3.5: Example 40 the function**

$$1/x^2$$

At $x = 0$, plugging in doesn't work, because division by zero is undefined. Applying the definition in terms of infinitesimals to the limit as $x$ approaches 0, we need to find out whether $1/(0 + dx)^2$ is finite for infinitesimal dx , and if so, whether it al- ways has the same standard part. But clearly $1/(0 + dx)^2 = dx^{-2}$ is always infinite, and we conclude that this limit is undefined.

## Example

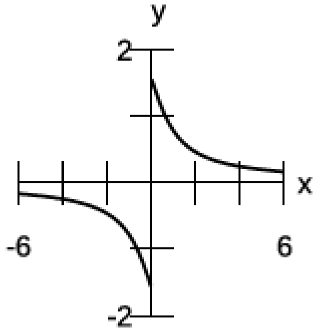


**Fig. 3.6:**

$$\tan^{-1}(1/x)$$

Investigate the limiting behavior of $f(x) = \tan^{-1}(1/x)$ as $x$ approaches 0.

Plugging in doesn't work, because division by zero is undefined.

In the definition of the limit in terms of infinitesimals, the first requirement is that $f(0 + dx)$ be finite for infinitesimal values of $dx$ . The graph makes this look plausible, and indeed we can prove that it is true by the transfer principle. For any real $x$ we have $-\pi/2 \le f(x) \le \pi/2$, and by the transfer principle this holds for the hyperreals as well, and therefore $f(0 + dx)$ is finite.

The second requirement is that the standard part of $f(0 + dx)$ have a uniquely defined value. The graph shows that we really have two cases to consider, one on the right side of the graph, and one on the left. Intuitively, we expect that the standard part of $f(0 + dx)$will equal $\pi/2$ for positive $dx$ , and $-\pi/2$ for negative, and thus the second part of the definition will not be satisfied. For a more formal proof, we can use the transfer principle. For real $x$ with $0 < x < 1$, for example, $f$ is always positive and greater than 1, so we conclude based on the transfer principle that $f(0 + dx) > 1$ for positive infinitesimal $dx$ . But on similar grounds we can be sure that $f(0 + dx) < -1$ when $dx$ is negative and infinitesimal. Thus the standard part of $f(0 + dx) > 1$ can have different values for different infinitesimal values of $dx$ , and we conclude that the limit is undefined.

In examples like this, we can define a kind of one-sided limit, notated like this:

$$\lim_{x \to 0} \tan^{-1} \frac{1}{x} = -\frac{\pi}{2}$$

$$\lim_{x \to 0} \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$$

where the notations $x \to 0^-$ and $x \to 0^+$ are to be read "as $x$ approaches zero from below," and "as x approaches zero from above."

## 3.3 L'Hopital's rule



Consider the limit

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Plugging in doesn't work, because we get 0/0. Division by zero is undefined, both in the real number system and in the hyperreals. A nonzero number divided by a small number gives a big number; a nonzero number divided by a very small number gives a very big number; and a nonzero number divided by an infinitesimal number gives an infinite number. On the other hand, dividing zero by zero means looking for a solution to the equation $0 = 0q$, where q is the result of the division. But any $q$ is a solution of this equation, so even speaking casually, it's not correct to say that 0/0 is infinite; it's not infinite, it's anything we like.

Since plugging in zero didn't work, let's try estimating the limit by plugging in a number for $x$ that's small, but not zero. On a calcula- tor,

$$\frac{\sin 0.00001}{0.00001} = 0.999999999983333$$

It looks like the limit is 1. We can confirm our conjecture to higher precision using Yacas's ability to do high-precision arithmetic:

```
N(Sin(10^-20)/10^-20,50)
0.99999999999999999
9999999999999999999
99998333333333
```

It's looking pretty one-ish. This is the idea of the Weierstrass definition of a limit: it seems like we can get an answer as close to 1 as we like, if we're willing to make $x$ as close to 0 as necessary. The graph helps to make this plausible.

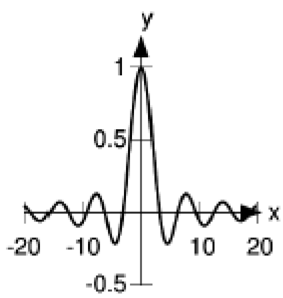


**Fig. 3.7: The graph of**

$$\sin\ x/x$$

The general idea here is that for small values of $x$, the small-angle approximation sin $x \approx x$ obtains, and as $x$ gets smaller and smaller, the approximation gets better and better, so $\sin\ x/x$ gets closer and closer to 1.

But we still haven't proved rigorously that the limit is exactly 1. Let's try using the definition of the limit in terms of infinitesimals.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x} &= st\left[\frac{\sin(0+dx)}{0+dx}\right] \\ &= st\left[\frac{dx+\ldots}{dx}\right]\end{aligned}$$

where we've used the identity $\sin(p+q) = \ \sin\ p\ \cos\ q + \ \sin\ q\ \cos\ p$, and . . . stands for terms of order $dx^2$. So

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x} &= st\left[1+\frac{\ldots}{dx}\right] \\ &= 1\end{aligned}$$

In fact, this limit is the same one we would use if we were evaluating the derivative of the sine function, applying the definition of the derivative as a limit.

We can check our work using Inf:

```
: (sin d)/d
1+(-0.16667)d^2+...
```

(The ... is where I've snipped trailing terms from the output.)

Our example involving the limit of sin $x/x$ is a special case of the following rule for calculating limits involving $0/0$:

## L'Hopital's rule (simplest form)

If $u$ and $v$ are functions with $u(a) = 0$ and $v(a) = 0$, the derivatives $\dot{u}(a)$ and $\dot{v}(a)$ are defined, and the derivative $\dot{v}(a) \neq 0$, then

$$\lim_{x\to a}\frac{u}{v} = \frac{\dot{u}(a)}{\dot{v}(a)}$$

Proof: Since $u(a) = 0$, and the derivative $du/dx$ is defined at $a$, $u(a+dx) = du$ is infinitesimal, and likewise for $v$. By the definition of the limit, the limit is the standard part of

$$\frac{u}{v} = \frac{du}{dv} = \frac{du/dx}{dv/dx}$$

where by assumption the numerator and denominator are both defined (and finite, because the derivative is defined in terms of the standard part). The standard part of a quotient like $p/q$ equals the quotient of the standard parts, provided that both $p$ and $q$ are finite (which we've established), and $q \neq 0$ (which is true by assumption). But the standard part of $du/dx$ is the definition of the derivative $\dot{u}$, and likewise for $dv/dx$, so this establishes the result.

We will generalize L'Hopital's rule in Generalizations of l'Hoˆpital's rule (Page 78).

By the way, the housetop accent on the "o" in l'Hopital means that in Old French it used to be spelled and pronounced "l'Hospital," but the "s" later became silent, so they stopped writing it. So yes, it is the same word as "hospital."

## Example

As remarked above, the example of $\lim\ x \to 0 \sin\ x/x$ is in some sense circular, since the limit is equivalent to the definition of the derivative of the sine function, so we already need to know the limit in order to evaluate the limit! As an example that isn't circular, let's evaluate

$$\lim_{X \to 0} \frac{\sin\ x}{x + x^3}$$

The derivative of the top is cos x , and the derivative of the bottom is 1. Evaluating these at $x = 0$ gives 1 and 1, so the answer is 1/1 = 1.

## Example

Evaluate

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

Taking the derivatives of the top and bottom, we find $e^x/1$, which equals 1 when evaluated at $x = 0$.

## Example

Evaluate

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 2x + 1}$$

Plugging in $x = 1$ fails, because both the top and the bottom are zero. Taking the derivatives of the top and bottom, we find $1/(2x - 2)$, which blows up to infinity when $x = 1$. To symbolize the fact that the limit is undefined, and undefined because it blows up to infinity, we write

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 2x + 1} = \infty$$

# 3.4 Another perspective on indeterminate forms



An expression like 0/0, called an indeterminate form, can be thought of in a different way in terms of infinitesimals. Suppose I tell you I have two infinitesimal numbers $d$ and $e$ in my pocket, and I ask you whether $d/e$ is finite, infinite, or infinitesimal. You can't tell, because $d$ and $e$ might not be infinitesimals of the same order of magnitude. For instance, if $e = 37d$, then $d/e = 1/37$ is finite; but if $e = d^2$, then $d/e$ is infinite; and if $d = e^2$, then d/e is infinitesimal. Acting this out with numbers that are small but not infinitesimal,

$$\frac{.001}{.037} = \frac{1}{37}$$
$$\frac{.001}{.000001} = 1000$$
$$\frac{.000001}{.001} = .001$$

On the other hand, suppose I tell you I have an infinitesimal number $d$ and a finite number $x$, and I ask you to speculate about $d/x$. You know for sure that it's going to be infinitesimal. Likewise, you can be sure that $x/d$ is infinite. These aren't indeterminate forms.

We can do something similar with infinite numbers. If $H$ and $K$ are both infinite, then $H - K$ is indeterminate. It could be infinite, for example, if $H$ was positive infinite and $K = H/2$. On the other hand, it could be finite if $H = K + 1$. Acting this out with big but finite numbers,

$$1000 - 500 = 500$$
$$1001 - 1000 = 1$$

## Example

If $H$ is a positive infinite number, is $\sqrt{H+1}-\sqrt{H-1}$ finite, infinite, infinitesimal, or indeterminate?

Trying it with a finite, big number, we have

$$\sqrt{1000001}-\sqrt{999999}$$
$$=1.00000000020373\times 10^{-3}$$

```
: H=1/d
d^-1
: sqrt(H+1)-sqrt(H-1)
d^1/2+0.125d^5/2+...
```

For convenience, the first line of input defines an infinite number $H$ in terms of the calculator's built-in infinitesimal $d$. The result has only positive powers of $d$, so it's clearly infinitesimal.

More rigorously, we can rewrite the expression as $\sqrt{H}\left(\sqrt{1+1/H}-\sqrt{1-1/H}\right)$. Since the derivative of the square root function $\sqrt{x}$ evaluated at $x=1$ is $1/2$, we can approximate this as

$$\sqrt{H}\left[1+\frac{1}{2H}+\ldots-\left(1-\frac{1}{2H}+\ldots\right)\right]$$

$$=\sqrt{H}\left[\frac{1}{H}+\ldots\right]$$

$$=\frac{1}{\sqrt{H}}$$

which is infinitesimal.

## 3.5 Limits at infinity



The definition of the limit in terms of infinitesimals extends immediately to limiting processes where x gets bigger and bigger, rather than closer and closer to some finite value. For example, the function $3+1/x$ clearly gets closer and closer to 3 as $x$ gets bigger and bigger. If $a$ is an infinite number, then the definition says that evaluating this expression at $a+dx$, where $dx$ is infinitesimal, gives a result whose standard part is 3. It doesn't matter that $a$ happens to be infinite, the definition still works. We also note that in this example, it doesn't matter what infinite number $a$ is; the limit equals 3 for any infinite $a$. We can write this fact as

$$\lim_{x\to\infty}\left(3+\frac{1}{x}\right)=3$$

where the symbol $\infty$ is to be interpreted as "nyeah nyeah, I don't even care what infinite number you put in here, I claim it will work out to 3 no matter what." The symbol $\infty$ is not to be interpreted as standing for any specific infinite number. That would be the type of fallacy that lay behind the bogus proof in Safe use of infinitesimals (Page 32) that 1 = 1/2, which assumed that all infinities had to be the same size.

A somewhat different example is the arctangent function. The arctangent of 1000 equals approximately 1.5698, and inputting bigger and bigger numbers gives answers that appear to get closer and closer to $\pi/2 \approx 1.5707$. But the arctangent of -1000 is approximately -1.5698, i.e., very close to $-\pi/2$. From these numerical observations, we conjecture that

$$\lim_{x \to a} \tan^{-1} x$$

equals $\pi/2$ for positive infinite $a$, but $-\pi/2$ for negative infinite $a$. It would not be correct to write

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{[wrong]}$$

because it does matter what infinite number we pick. Instead we write

$$\lim_{x \to +\infty} \tan^{-1} x = \frac{\pi}{2}$$
$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Some expressions don't have this kind of limit at all. For example, if you take the sines of big numbers like a thousand, a million, etc., on your calculator, the results are essentially random numbers lying between 1 and 1. They don't settle down to any particular value, because the sine function oscillates back and forth forever. To prove formally that $\lim_{x\to+\infty} \sin x$ is undefined, consider that the sine function, defined on the real numbers, has the property that you can always change its result by at least 0.1 if you add either 1.5 or 1.5 to its input. For example, $\sin(.8) \approx 0.717$, and $\sin(.8-1.5) \approx 0.644$. Applying the transfer principle to this statement, we find that the same is true on the hyperreals. Therefore there cannot be any value l that differs infinitesimally from $\sin a$ for all positive infinite values of $a$.

Often we're interested in finding the limit as $x$ approaches infinity of an expression that is written as an indeterminate form like $H/K$, where both $H$ and $K$ are infinite.

## Example

Evaluate the limit

$$\lim_{x \to \infty} \frac{2x + 7}{x + 8686}$$

Intuitively, if $x$ gets large enough the constant terms will be negligible, and the top and bottom will be dominated by the $2x$ and $x$ terms, respectively, giving an answer that approaches 2.

One way to verify this is to divide both the top and the bottom by $x$, giving

$$\frac{2 + \frac{7}{x}}{1 + \frac{8686}{x}}$$

If x is infinite, then the standard part of the top is 2, the standard part of the bottom is 1, and the standard part of the whole thing is therefore 2.

Another approach is to use l'Hopital's rule. The derivative of the top is 2, and the derivative of the bottom is 1, so the limit is $2/1 = 2$.

# 3.6 Generalizations of l'Hoˆpital's rule



Mathematical theorems are some- times like cars. I own a Honda Fit that is about as bare-bones as you can get these days, but persuading a dealer to sell me that car was like pulling teeth. The sales- man was absolutely certain that any sane customer would want to pay an extra $1,800 for such crucial amenities as floor mats and a chrome tailpipe. L'Hˆopital's rule in its most general form is a much fancier piece of machinery than the stripped down model described in L'Hopital's rule (Page 72). The price you pay for the deluxe model is that the proof becomes much more complicated than the one-liner that sufficed for the simple version.

## 3.6.1 Multiple applications of the rule



In the following example, we have to use l'H^opital's rule twice before we get an answer.

## Example

Evaluate

$$\lim_{x \to \pi} \frac{1 + cosx}{(x = \pi)^2}$$

Applying l'Hopital's rule gives

$$\frac{-sinx}{2(x - \pi)}$$

which still produces 0/0 when we plug in $x = \pi$. Going again, we get

$$\frac{-cosx}{2} = \frac{1}{2}$$

The reason that this always works is outlined in Proofs of the generalizations of l'Hoˆ pital's rule (Page 179).

### 3.6.2 The indeterminate form ∞/∞



Consider an example like this:

$$\lim_{x \to 0} \frac{1 + 1/x}{1 + 2/x}$$

This is an indeterminate form like $\infty/\infty$ rather than the 0/0 form for which we've already proved l'Hopital's rule. As proved in Proofs of the generalizations of l'Hoˆ pital's rule (Page 179), l'Hopital's rule applies to examples like this as well.

## Example

Evaluate

$$\lim_{x \to 0} \frac{1 + 1/x}{1 + 2/x}$$

Both the numerator and the denominator go to infinity. Differentiation of the top and bottom gives $(-x^{-2})/(-2x^{-2}) = 1/2$. We can see that the reason the rule worked was that (1) the constant terms were irrelevant because they become negligible as the $1/x$ terms blow up; and (2) differentiating the blowing-up $1/x$ terms makes them into the same $x^{-2}$ on top and bottom, which cancel.

Note that we could also have gotten this result without l'Hopital's rule, simply by multiplying both the top and the bottom of the original expression by $x$ in order to rewrite it as $(x + 1x)/(x + 2)$.

### 3.6.3 Limits at infinity



It is straightforward to prove a variant of l'Hopital's rule that allows us to do limits at infinity. The general proof is left as an exercise (Problem 3.8 (Page 82)). The result is that l'H^opital's rule is equally valid when the limit is at $\pm\infty$ rather than at some real number $a$.

**Example**

Evaluate

$$\lim_{\to\infty} \frac{2+7}{x+8686}$$

We could use a change of variable to make this into Example (Page 69), which was solved using an ad hoc and multiple-step procedure. But having established the more general form of l'Hoˆ pital's rule, we can do it in one step. Differentiation of the top and bot- tom produces

$$\lim_{x\to\infty} \frac{2+7}{X+8686} = \frac{2}{1} = 1$$

## 3.7 Problems

## 3.8 Problem 3.1



(a) Prove, using the Weier- strass definition of the limit, that if limx→a f (x) = F and limx→a g(x) = G both exist, them exactly, and check your result by numerical approximation.

Solutions for chapter 3 (Page 206)

## 3.9 Problem 3.2



Sketch the graph of the function $e^{-1/x}$, and evaluate the following four limits:

$$\lim_{x\to 0+} e^{-1/x}$$
$$\lim_{x\to 0-} e^{-1/x}$$
$$\lim_{x\to +\infty} e^{-1/x}$$
$$\lim_{x\to -\infty} e^{-1/x}$$

Solutions for chapter 3 (Page 206)

## 3.10 Problem 3.3



Verify the following limits.

$$\lim_{s\to 1}\frac{s^3-1}{s-1}=3$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\frac{1}{2}$$

$$\lim_{x\to\infty}\frac{5x^2-2x}{x}=\infty$$

$$\lim_{n\to\infty}\frac{n(n+1)}{(n+2)(n+3)}=1$$

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=\frac{a}{d}$$

Solutions for chapter 3 (Page 206)

## 3.11 Problem 3.4



Evaluate

$$\lim_{x\to 0}\frac{x\ \cos\ x}{1-2^x}$$

exactly, and check your result by numerical approximation.

Solutions for chapter 3 (Page 206)

## 3.12 Problem 3.5



Amy is asked to evaluate

$$\lim_{x\to 0}\frac{e^x}{x}$$

She applies l'Hopital's rule, differentiating top and bottom to find $1/e^x$, which equals 1 when she plugs in $x=0$. What is wrong with her reasoning?

Solutions for chapter 3 (Page 206)

## 3.13 Problem 3.6



Evaluate

$$\lim_{u \to 0} \frac{u^2}{e^u + e^{-u} - 2}$$

exactly, and check your result by numerical approximation.

Solutions for chapter 3 (Page 206)

## 3.14 Problem 3.7



Evaluate

$$\lim_{t \to \pi} \frac{\sin t}{t - \pi}$$

exactly, and check your result by numerical approximation.

Solutions for chapter 3 (Page 206)

## 3.15 Problem 3.8



Prove a form of l'Hopital's rule stating that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

is equal to the limit of $f'/g'$ at infinity. Hint: change to some new variable $u$ such that $x \to \infty$ corresponds to $u \to 0$.

Solutions for chapter 3 (Page 206)

## 3.16 Problem 3.9



Prove that the linear function $y = ax + b$, where $a$ and $b$ are real, is continuous, first using the definition of continuity in terms of infinitesimals, and then using the definition in terms of the Weier- strass limit.

Solutions for chapter 3 (Page 206)

# Chapter 4 Integration

## 4.1 Definite and indefinite integrals



Because any formula can be differentiated symbolically to find an- other formula, the main motivation for doing derivatives numerically would be if the function to be differentiated wasn't known in symbolic form. A typical example might be a two-person network computer game, in which player A's computer needs to figure out player B's velocity based on knowledge of how her position changes over time. But in most cases, it's numerical integration that's interesting, not numerical differentiation.

As a warm-up, let's see how to do a running sum of a discrete function using Yacas. The following program computes the sum $1+2+\ldots+100$ discussed to in Change in discrete steps (Page 1). Now that we're writing real computer programs with Yacas, it would be a good idea to enter each program into a file before trying to run it. In fact, some of these examples won't run properly if you just start up Yacas and type them in one line at a time. If you're using Adobe Reader to read this book, you can do `Tools>Basic>Select`, select the program, copy it into a file, and then edit out the line numbers.

**Example**

```
1  n := 1;
2  sum := 0;
3  While (n<=100) [
4  sum := sum+n;
5  n := n+1;
6  ];
7  Echo(sum)
```

The semicolons are to separate one instruction from the next, and they become necessary now that we're doing real programming. Line 1 of this program defines the variable n, which will take on all the values from 1 to 100. Line 2 says that we haven't added anything up yet, so our running sum is zero so far. Line 3 says to keep on repeating the instructions inside the square brackets until n goes past 100. Line 4 updates the running sum, and line 5 updates the value of n. If you've never done any programming before, a statement like `n:=n+1` might seem like nonsense — how can a number equal itself plus one? But that's why we use the := symbol; it says that we're redefining `n`, not stating an equation. If `n` was previously 37, then after this statement is executed, n will be redefined as 38. To run the program on a Linux computer, do this (assuming you saved the pro- gram in a file named `sum.yacas`):

```
yacas -pc sum.yacas
5050
```

Here the % symbol is the computer's prompt. The result is 5,050, as expected. One way of stating this result is

$$\sum_{n=1}^{100} n = 5050$$

The capital Greek letter $\sum$, sigma, is used because it makes the "s" sound, and that's the first sound in the word "sum." The $n = 1$ below the sigma says the sum starts at 1, and the 100 on top says it ends at 100. The $n$ is what's known as a dummy variable: it has no meaning outside the context of the sum. Figure 4.1 shows the graphical interpretation of the sum: we're adding up the areas of a series of rectangular strips. (For clarity, the figure only shows the sum going up to 7, rather than 100.)

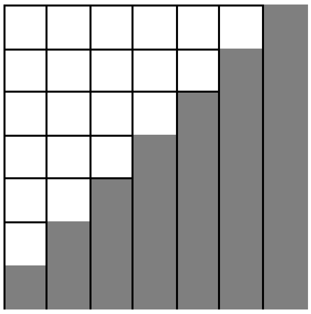

**Fig. 4.1: Graphical interpretation of the sum 1+2+...+7**

Now how about an integral? Figure 4.2 shows the graphical interpretation of what we're trying to do: find the area of the shaded triangle. This is an example we know how to do symbolically, so we can do it numerically as well, and check the answers against each other. Symbolically, the area is given by the integral. To integrate the function $\dot{x}(t) = t$, we know we need some function with a $t^2$ in it, since we want something whose derivative is $t$, and differentiation reduces the power by one. The derivative of $t^2$ would be $2t$ rather than $t$, so what we want is $x(t) = t^2/2$. Let's compute the area of the triangle that stretches along the $t$ axis from 0 to 100: $x(100) = 100^2/2 = 5000$.

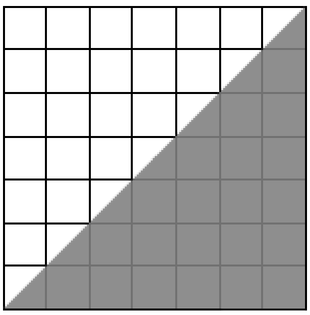

**Fig. 4.2: Graphical interpretation of the integral of the function** $\dot{x}(t) = t$

Figure 4.3 shows how to accomplish the same thing numerically. We break up the area into a whole bunch of very skinny rectangles. Ideally, we'd like to make the width of each rectangle be an infinitesimal number $dx$, so that we'd be adding up an infinite number of infinitesimal areas. In reality, a computer can't do that, so we divide up the interval from $t = 0$ to $t = 100$ into $H$ rectangles, each with finite width $dt = 100/H$ . Instead of making H infinite, we make it the largest number we can without making the computer take too long to add up the areas of the rectangles.

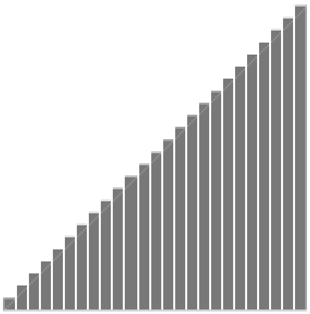

**Fig. 4.3: Approximating the integral numerically.**

## Example

```
tmax := 100;
H := 1000;
dt := tmax/H;
sum := 0;
t := 0;
While (t<=tmax) [
 sum := N(sum+t*dt);
 t := N(t+dt);
];
 Echo(sum);
```

In Example (Page 85), we split the interval from $t = 0$ to 100 into $H = 1000$ small intervals, each with width $dt = 0.1$. The result is 5,005, which agrees with the symbolic result to three digits of precision. Changing $H$ to 10,000 gives 5, 000.5, which is one more digit. Clearly as we make the number of rectangles greater and greater, we're converging to the correct result of 5,000.

In the Leibniz notation, the thing we've just calculated, by two different techniques, is written like this:

$$\int_0^{100} t dt = 5,000$$

It looks a lot like the $\sum$ notation, with the $\sum$ replaces by a flattened out letter "S." The $t$ is a dummy variable. What I've been casually referring to as an integral is re- ally two different but closely related things, known as the definite integral and the indefinite integral.

## Definition of the indefinite integral

If $\dot{x}$ is a function, then a function $x$ is an indefinite integral of $\dot{x}$ if, as implied by the notation, $dx/dt = \dot{x}$.

Interpretation: Doing an indefinite integral means doing the opposite of differentiation. All the possible indefinite integrals are the same function except for an additive constant.

## Example

Find the indefinite integral of the function $\dot{x}(t) = t$.

Any function of the form

$$x(t) = t^2/2 + c$$

where $c$ is a constant, is an indefinite integral of this function, since its derivative is $t$.

## Definition of the definite integral

If $\dot{x}$ is a function, then the definite integral of $\dot{x}$ from a to b is defined as

$$\int_a^b \dot{x}(t)dt$$
$$= \lim_{H\to\infty} \sum_{i=0}^{H} \dot{x}(a + i\Delta t)\Delta t$$

where

$$\Delta t = (b-a)/H$$

Interpretation: What we're calculating is the area under the graph of $\dot{x}$, from $a$ to $b$. (If the graph dips below the $t$ axis, we interpret the area between it and the axis as a negative area.) The thing inside the limit is a calculation like the one done in Example (Page 85), but generalized to $a \neq 0$. If $H$ was infinite, then $\Delta t$ would be an infinitesimal number $dt$.

## 4.2 The fundamental theorem of calculus



### The fundamental theorem of calculus

Let $x$ be an indefinite integral of $\dot{x}$, and let $\dot{x}$ be a continuous function (one whose graph is a single connected curve). Then

$$\int_a^b \dot{x}(t)dt = x(b) - x(a)$$

The fundamental theorem is proved in Proof of the fundamental theorem of calculus (Page 182). The idea it expresses is that integration and differentiation are inverse operations. That is, integration undoes differentiation, and differentiation undoes integration.

### Example

Interpret the definite integral

$$\int_1^2 \frac{1}{t}dt$$

graphically; then evaluate it it both symbolically and numerically, and check that the two results are consistent.

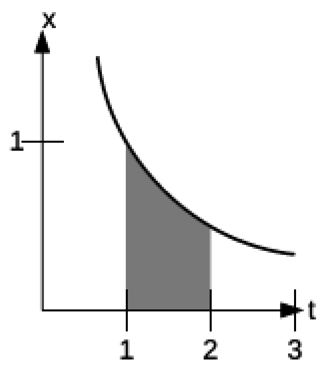


**Fig. 4.4:** The definite integral

$$\int_1^2 (1/t)dt$$

Figure 4.4 shows the graphical interpretation. The numerical calculation requires a trivial variation on the program from Example (Page 85):

```
a := 1;
b := 2;
H := 1000;
dt := (b-a)/H;
sum := 0;
t := a;
While (t<=b) [
 sum := N(sum+(1/t)*dt);
 t := N(t+dt);
];
Echo(sum);
```

The result is 0.693897243, and increasing $H$ to 10,000 gives 0.6932221811, so we can be fairly confident that the result equals 0.693, to 3 decimal places.

Symbolically, the indefinite integral is $x = \ln t$. Using the fundamental theorem of calculus, the area is $\ln 2 - \ln 1 \approx 0.693147180559945$.

Judging from the graph, it looks plausible that the shaded area is about 0.7.

This is an interesting example, be- cause the natural log blows up to negative infinity as $t$ approaches 0, so it's not possible to add a constant onto the indefinite integral and force it to be equal to 0 at $t = 0$. Nevertheless, the fundamental theorem of calculus still works.

## 4.3 Properties of the integral



Let $f$ and $g$ be two functions of $x$, and let $c$ be a constant. We already know that for derivatives,

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

and

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

But since the indefinite integral is just the operation of undoing a derivative, the same kind of rules must hold true for indefinite integrals as well:

$$\int (f+g)dx = \int f\ dx + \int g\ dx$$

and

$$\int (cf)dx = c\int f\ dx$$

And since a definite integral can be found by plugging in the upper and lower limits of integration into the indefinite integral, the same properties must be true of definite integrals as well.

**Example**

Evaluate the indefinite integral

$$\int (x + 2\sin\ x)dx$$

Using the additive property, the integral becomes

$$\int xdx + \int 2\sin\ x\ dx$$

Then the property of scaling by a constant lets us change this to

$$\int xdx + 2\int \sin\ x\ dx$$

We need a function whose derivative is $x$, which would be $x^2/2$, and one whose derivative is $\sin\ x$, which must be $-\cos\ x$, so the result is

$$\frac{1}{2}x^2 - 2\cos\ x + c$$

## 4.4 Applications Averages



In the story of Gauss's problem of adding up the numbers from 1 to100, one interpretation of the result, 5,050, is that the average of all the numbers from 1 to 100 is 50.5. This is the ordinary definition of an average: add up all the things you have, and divide by the number of things. (The result in this example makes sense, because half the numbers are from 1 to 50, and half are from 51 to 100, so the average is half-way between 50 and 51.)

Similarly, a definite integral can also be thought of as a kind of aver- age. In general, if $y$ is a function of $x$, then the average, or mean, value of $y$ on the interval from $x = a$ to $b$ can be defined as

$$\bar{y} = \frac{1}{b-a}\int_b^a y\ dx$$

In the continuous case, dividing by b − a accomplishes the same thing as dividing by the number of things in the discrete case.

## Example

Show that the definition of the average makes sense in the case where the function is a constant.If $y$ is a constant, then we can take it outside of the integral, so

$$\begin{aligned}\bar{y} &= \frac{1}{b-a} y \int_a^b 1\ dx \\ &= \frac{1}{b-a} yx \mid_a^b \\ &= \frac{1}{b-a} y(b-a) \\ &= y\end{aligned}$$

## Example

Find the average value of the function $y = x^2$ for values of $x$ ranging from 0 to 1.

$$\begin{aligned}\bar{y} &= \frac{1}{1-0} \int_0^1 x^2 dx \\ &= \frac{1}{3} x^3 \mid_0^1 \\ &= \frac{1}{3}\end{aligned}$$

## The mean value theorem

If the continuous function $y(x)$ has the average value $\bar{y}$ on the interval from $x = a$ to $b$, then $y$ attains its average value at least once in that interval, i.e., there exists $\xi$ with $a < \xi < b$ such that $y(\xi) = \bar{y}$.

The mean value theorem is proved in Proof of the mean value theorem (Page 187). The special case in which $\bar{y} = 0$ is known as Rolle's theorem.

## Example

Verify the mean value theorem for $y = x^2$ on the interval from 0 to 1.

The mean value is 1/3, as shown in Example (Page 90). This value is achieved at $x = \sqrt{1/3} = 1/\sqrt{3}$, which lies between 0 and 1.

### 4.4.1 Work



In physics, work is a measure of the amount of energy transferred by a force; for example, if a horse sets a wagon in motion, the horse's force on the wagon is putting some energy of motion into the wagon. When a force $F$ acts on an object that moves in the direction of the force by an infinitesimal distance $dx$, the infinitesimal work done is $dW = Fdx$. Integrating both sides, we have $W = \int_a^b Fdx$, where the force may depend on $x$, and $a$ and $b$ represent the initial and final positions of the object.

**Example**

A spring compressed by an amount $x$ relative to its relaxed length provides a force $F = kx$. Find the amount of work that must be done in order to compress the spring from $x = 0$ to $x = a$. (This is the amount of energy stored in the spring, and that energy will later be released into the toy bullet.)

$$\begin{aligned} W &= \int_0^a Fdx \\ &= \int_0^a kxdx \\ &= \frac{1}{2}kx^2 \; |_0^a \\ &= \frac{1}{2}ka^2 \end{aligned}$$

The reason $W$ grows like $a^2$, not just like $a$, is that as the spring is com- pressed more, more and more effort is required in order to compress it.

### 4.4.2 Probability



Mathematically, the probability that something will happen can be specified with a number ranging from 0 to 1, with 0 representing impossibility and 1 representing certainty. If you flip a coin, heads and tails both have probabilities of 1/2. The sum of the probabilities of all the possible outcomes has to have probability 1. This is called normalization.

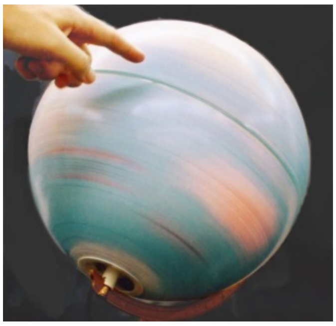

**Fig. 4.5: Normalization** the probability of picking land plus the probability of picking water adds up to 1.

So far we've discussed random processes having only two possible outcomes: yes or no, win or lose, on or off. More generally, a random process could have a result that is a number. Some processes yield integers, as when you roll a die and get a result from one to six, but some are not restricted to whole numbers, e.g., the height of a human being, or the amount of time that a uranium-238 atom will exist before undergoing radioactive decay. The key to handling these continuous random variables is the concept of the area under a curve, i.e., an integral.

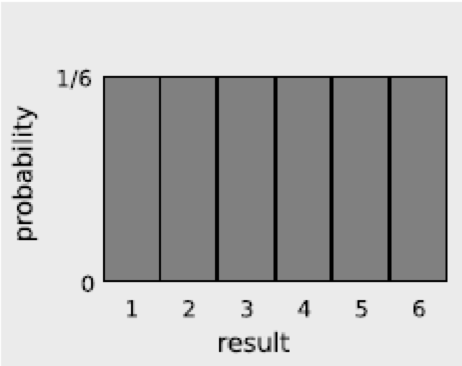


**Fig. 4.6: Probability distribution for the result of rolling a single die.**

Consider a throw of a die. If the die is "honest," then we expect all six values to be equally likely. Since all six probabilities must add up to 1, then probability of any particular value coming up must be 1/6. We can summarize this in a graph, f. Areas under the curve can be interpreted as total probabilities. For instance, the area under the curve from 1 to 3 is 1/6+1/6+1/6 = 1/2, so the probability of getting a result from 1 to 3 is 1/2. The function shown on the graph is called the probability distribution.

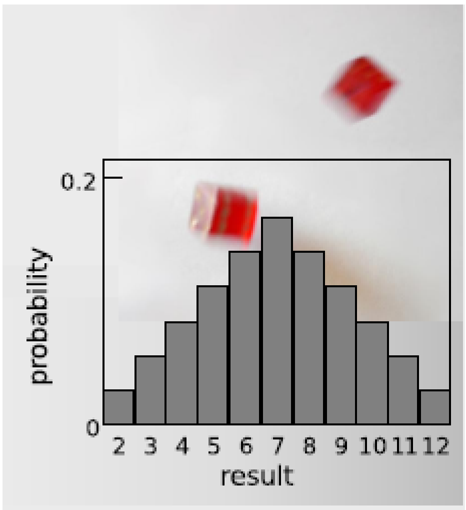


**Fig. 4.7: Rolling two dice and adding them up.**

Figure 4.7 shows the probabilities of various results obtained by rolling two dice and adding them together, as in the game of craps. The probabilities are not all the same. There is a small probability of getting a two, for example, be- cause there is only one way to do it, by rolling a one and then another one. The probability of rolling a seven is high because there are six different ways to do it: 1+6, 2+5, etc.

If the number of possible outcomes is large but finite, for example the number of hairs on a dog, the graph would start to look like a smooth curve rather than a ziggurat.

What about probability distributions for random numbers that are not integers? We can no longer make a graph with probability on the y axis, because the probability of getting a given exact number is typically zero. For instance, there is zero probability that a per- son will be exactly 200 cm tall, since there are infinitely many possible results that are close to 200 but not exactly two, for example 199.99999999687687658766. It doesn't usually make sense, therefore, to talk about the probability of a single numerical result, but it does make sense to talk about the probability of a certain range of results. For instance, the probability that a randomly chosen person will be more than 170 cm and less than 200 cm in height is a perfectly reasonable thing to discuss. We can still summarize the probability in- formation on a graph, and we can still interpret areas under the curve as probabilities.

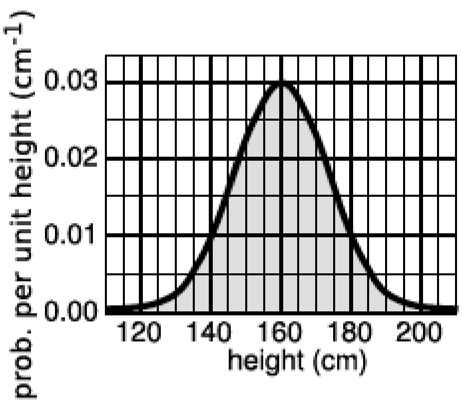


**Fig. 4.8: A probability distribution for human height.**

But the $y$ axis can no longer be a unitless probability scale. In the example of human height, we want the $x$ axis to have units of meters, and we want areas under the curve to be unitless probabilities. The area of a single square on the graph paper is then

$$(\text{unitless area of a square}) = (\text{width of square with distance units}) \times (\text{height of square})$$

If the units are to cancel out, then the height of the square must evidently be a quantity with units of inverse centimeters. In other words, the $y$ axis of the graph is to be interpreted as probability per unit height, not probability.

Another way of looking at it is that the $y$ axis on the graph gives a derivative, $dP/dx$: the infinitesimally small probability that $x$ will lie in the infinitesimally small range covered by $dx$.

## Example

A computer language will typically have a built-in subroutine that produces a fairly random number that is equally likely to take on any value in the range from 0 to 1. If you take the absolute value of the difference between two such numbers, the probability distribution is of the form $dP/dx = k(1-x)$. Find the value of the constant $k$ that is required by normalization.

$$\begin{aligned} 1 &= \int_0^1 (1-x)dx \\ &= kx - \frac{1}{2}kx^2 \ |_0^1 \\ &= k - k/2 \\ k &= 2 \end{aligned}$$

## Self-Check.

Compare the number of people with heights in the range of 130-135 cm to the number in the range 135-140.

Answers to self-checks for chapter 4 (Page 189)

**Fig. 4.9:** The average can be interpreted as the balance point of the probability distribution.

When one random variable is related to another in some mathematical way, the chain rule can be used to relate their probability distributions.

## Example

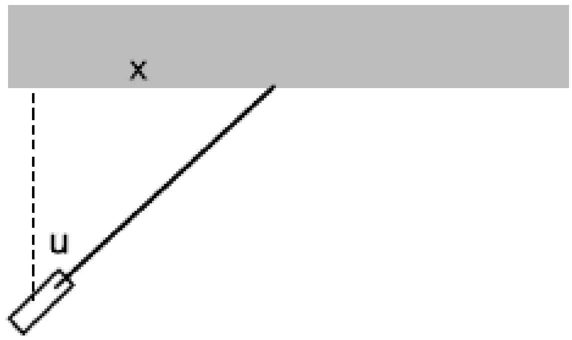


**Fig. 4.10:**

A laser is placed one meter away from a wall, and spun on the ground to give it a random direction, but if the angle $u$ shown in Figure 4.10 doesn't come out in the range from 0 to $\pi/2$, the laser is spun again until an angle in the desired range is obtained. Find the probability distribution of the distance $x$ shown in the figure. The derivative $\mathrm{d}\tan^{-1} z/dz = 1/(1+z^2)$ will be required (see Example (Page 106)).

Since any angle between 0 and $\pi/2$ is equally likely, the probability distribution $dP/du$ must be a constant, and normalization tells us that the constant must be $dP/du = 2/\pi$.

The laser is one meter from the wall, so the distance $x$, measured in meters, is given by $x = \tan u$. For the probability distribution of $x$, we have

$$\begin{aligned} \frac{dP}{dx} &= \frac{dP}{du} \cdot \frac{du}{dx} \\ &= \frac{2}{\pi} \cdot \frac{\mathrm{d}\tan^{-1} x}{dx} \\ &= \frac{2}{\pi(1+x^2)} \end{aligned}$$

Note that the range of possible values of $x$ theoretically extends from 0 to infinity. Problem 6.7 (Page 126) deals with this.

If the next Martian you meet asks you, “How tall is an adult hu- man?,” you will probably reply with a statement about the average human height, such as “Oh, about 5 feet 6 inches.” If you wanted to explain a little more, you could say, “But that’s only an average. Most people are somewhere between 5 feet and 6 feet tall.” Without bothering to draw the relevant bell curve for your new extraterrestrial acquaintance, you’ve summarized the relevant information by giving an average and a typical range of variation. The average of a probability distribution can be defined geometrically as the horizontal position at which it could be balanced if it was constructed out of cardboard, i. This is a different way of working with averages than the one we did earlier. Before, had a graph of $y$ versus $x$, we implicitly assumed that all values of $x$ were equally likely, and we found an average value of $y$. In this new method using probability distributions, the variable we're averaging is on the $x$ axis, and the $y$ axis tells us the relative probabilities of the various $x$ values.

For a discrete-valued variable with $n$ possible values, the average would be

$$\bar{x} = \sum_{i=0}^{n} xP(x)$$

and in the case of a continuous variable, this becomes an integral,

$$\bar{x} = \int_a^b x\frac{dP}{dx}dx$$

## Example

For the situation described in Example (Page 94), find the average value of $x$.

$$\begin{aligned}
\bar{x} &= \int_0^1 x\frac{dP}{dx}dx \\
&= \int_0^1 x \cdot 2(1-x)dx \\
&= 2\int_0^1 (x - x^2)dx \\
&= 2\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

Sometimes we don't just want to know the average value of a certain variable, we also want to have some idea of the amount of variation above and below the average. The most common way of measuring this is the *standard deviation*, defined by

$$\sigma = \sqrt{\int_a^b (x-\bar{x})^2\frac{dP}{dx}dx}$$

The idea here is that if there was no variation at all above or below the average, then the quantity $(x - \bar{x})$ would be zero whenever $dP/dx$ was nonzero, and the standard deviation would be zero. The reason for taking the square root of the whole thing is so that the result will have the same units as $x$.

## Example

For the situation described in exam- ple 59, find the standard deviation of $x$ .

The square of the standard deviation is

$$\begin{aligned}\sigma^2 &= \int_0^1 (x-\bar{x})^2 \frac{dP}{dx} dx \\ &= \int_0^1 (x-1/3)^2 \cdot 2(1-x) dx \\ &= 2\int_0^1 \left(-x^3+\frac{5}{3}x^2-\frac{7}{9}x+\frac{1}{9}\right) dx \\ &= \frac{1}{18}\end{aligned}$$

so the standard deviation is

$$\begin{aligned}\sigma &= \frac{1}{\sqrt{18}} \\ &\approx 0.236\end{aligned}$$

## 4.5 Problems

## 4.6 Problem 4.1



Write a computer program similar to the one in Example (Page 87) to evaluate the definite integral

$$\int_0^1 e^{x^2}$$

## 4.7 Problem 4.2



Evaluate the integral

$$\int_0^{2\pi} \sin x dx$$

and draw a sketch to explain why your result comes out the way it does.

Solutions for chapter 4 (Page 208)

## 4.8 Problem 4.3



Sketch the graph that represents the definite integral

$$\int_0^2 (-x^2 + 2x)dx$$

and estimate the result roughly from the graph. Then evaluate the integral exactly, and check against your estimate.

Solutions for chapter 4 (Page 208)

## 4.9 Problem 4.4



Make a rough guess as to the average value of $\sin x$ for $0 < x < \pi$, and then find the exact result and check it against your guess.

Solutions for chapter 4 (Page 208)

## 4.10 Problem 4.5



Show that the mean value theorem's assumption of continuity is necessary, by exhibiting a discontinuous function for which the theorem fails.

Solutions for chapter 4 (Page 208)

## 4.11 Problem 4.6



Show that the fundamental theorem of calculus's assumption of continuity for $\dot{x}$ is necessary, by exhibiting a discontinuous function for which the theorem fails.

Solutions for chapter 4 (Page 208)

## 4.12 Problem 4.7



Sketch the graphs of $y = x^2$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$. Graphically, what relationship should exist between the integrals $\int_0^1 x^2 dx$ and $\int_0^1 \sqrt{x} dx$? Compute both integrals, and verify that the results are related in the expected way.

## 4.13 Problem 4.8



Evaluate $\int \sqrt{bx\sqrt{x}} dx$, where $b$ is a constant.

Solutions for chapter 4 (Page 208)

## 4.14 Problem 4.9



In a gasoline-burning car engine, the exploding air-gas mixture makes a force on the piston, and the force tapers off as the piston expands, allowing the gas to expand. (a) In the approximation $F = k/x$, where $x$ is the position of the piston, find the work done on the piston as it travels from $x = a$ to $x = b$, and show that the result only depends on the ratio $b/a$. This ratio is known as the compression ratio of the engine. (b) A better approximation, which takes into account the cooling of the air-gas mixture as it expands, is $F = kx^{-1.4}$. Compute the work done in this case.

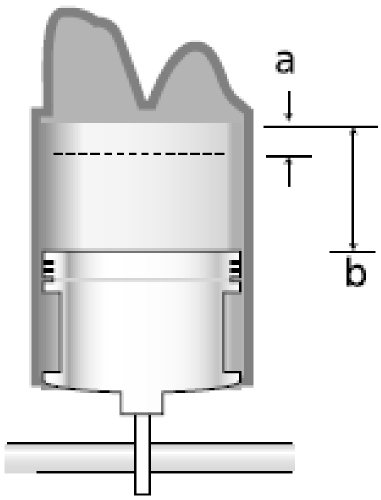


**Fig. 4.11:**

## 4.15 Problem 4.10



A certain variable $x$ varies randomly from -1 to 1, with probability distribution $dP/dx = k(1-x^2)$.

a. Determine $k$ from the requirement of normalization.

b. Find the average value of x.

c. Find its standard deviation.

## 4.16 Problem 4.11



Suppose that we've already established that the derivative of an odd function is even, and vice versa. (See Problem 2.30 (Page 58)) Something similar can be proved for integration. However, the following is not quite right. *Let* $f$ *be even, and let* $g=\int f(x)dx$ *be its indefinite integral. Then by the fundamental theorem of calculus,* $f$ *is the derivative of* $g$*. Since we've already established that the derivative of an odd function is even, we conclude that* $g$ *is odd.* Find all errors in the proof.

Solutions for chapter 4 (Page 208)

## 4.17 Problem 4.12



A perfectly elastic ball bounces up and down forever, always coming back up to the same height h. Find its average height.

## 4.18 Problem 4.13



The figure shows a curve with a tangent line segment of length 1 that sweeps around it, forming a new curve that is usually outside the old one. Prove Holditch's theorem, which states that the new curve's area differs from the old one's by $\pi$. (This is an example of a result that is much more difficult to prove without making use of infinitesimals.)

# Chapter 5 Techniques

## 5.1 Newton's method



In the 1958 science fiction novel **Have Space Suit - Will Travel**, by Robert Heinlein, Kip is a high school student who wants to be an engineer, and his father is trying to convince him to stretch himself more if he wants to get any- thing out of his education:

*"Why did Van Buren fail of re- election? How do you extract the cube root of eighty-seven?"* *Van Buren had been a president; that was all I remembered. But I could answer the other one. "If you want a cube root, you look in a table in the back of the book."*

*Dad sighed. "Kip, do you think that table was brought down from on high by an archangel?"*

We no longer use tables to compute roots, but how does a pocket calculator do it? A technique called Newton's method allows us to calculate the inverse of any function efficiently, including cases that aren't preprogrammed into a calculator. In the example from the novel, we know how to calculate the function $y = x^3$ fairly accurately and quickly for any given value of $x$, but we want to turn the equation around and find $x$ when $y = 87$. We start with a rough mental guess: since $4^3 = 64$ is a little too small, and $5^3 = 125$ is much too big, we guess $x \approx 4.3$. Testing our guess, we have $4.3^3 = 79.5$. We want $y$ to get bigger by 7.5, and we can use calculus to find approximately how much bigger $x$ needs to get in order to accomplish that:

$$\begin{aligned} \Delta x &= \frac{\Delta x}{\Delta y}\Delta y \\ &\approx \frac{dx}{dy}\Delta y \\ &= \frac{\Delta y}{dy/dx} \\ &= \frac{\Delta y}{3x^2} \\ &= \frac{\Delta y}{3x^2} \\ &= 0.14 \end{aligned}$$

Increasing our value of $x$ to 4.3 +0.14 = 4.44, we find that $4.44^3$ =87.5 is a pretty good approximation to 87. If we need higher precision, we can go through the process again with $\Delta y = -0.5$, giving

$$\Delta x \approx \frac{\Delta y}{3x^2}$$
$$= 0.14$$
$$x = 4.43$$
$$x^3 = 86.9$$

This second iteration gives an excellent approximation.

## Example

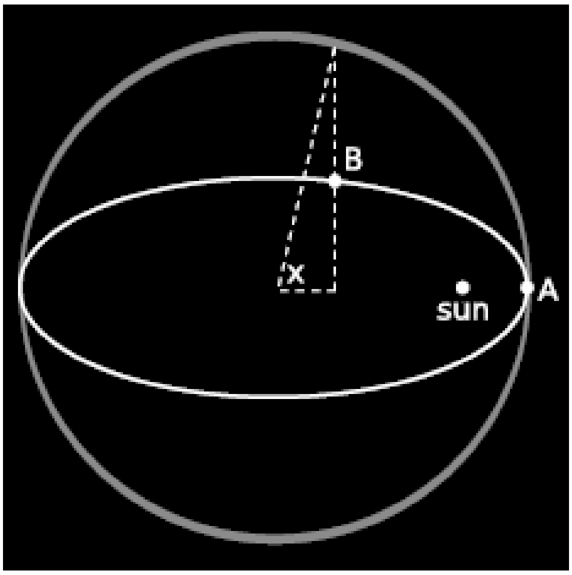


**Fig. 5.1:**

Figure 5.1 shows the astronomer Johannes Kepler's analysis of the motion of the planets. The ellipse is the orbit of the planet around the sun. At $t = 0$, the planet is at its closest approach to the sun, A. At some later time, the planet is at point B. The angle x (measured in radians) is defined with reference to the imaginary circle encompassing the orbit. Kepler found the equation

$$2\pi \frac{t}{T} = x - e \sin x$$

where the period, $T$, is the time required for the planet to complete a full orbit, and the eccentricity of the ellipse, $e$, is a number that measures how much it differs from a circle. The relationship is complicated because the planet speeds up as it falls inward toward the sun, and slows down again as it swings back away from it.

The planet Mercury has $e = 0.206$. Find the angle $x$ when Mercury has completed 1/4 of a period.

We have

$$y = x - (0.206) \sin x$$

and we want to find $x$ when $y = 2\pi/4 = 1.57$. As a first guess, we try $x = \pi/2$ (90 degrees), since the eccentricity of Mercury's orbit is actually much smaller than the example shown in the figure, and therefore the planet's speed doesn't vary all that much as it goes around the sun. For this value of $x$ we have $y = 1.36$, which is too small by 0.21.

$$\begin{aligned}\Delta x &\approx \frac{\Delta y}{dy/dx} \\ &= \frac{0.21}{1-(0.206)\cos\ x} \\ &= 0.21\end{aligned}$$

(The derivative $dy/dx$ happens to be 1 at $x = \pi/2$.) This gives a new value of $x$ , 1.57+.21=1.78. Testing it, we have $y = 1.58$, which is correct to within rounding errors after only one iteration. (We were only supplied with a value of e accurate to three significant figures, so we can't get a result with precision better than about that level.)

## 5.2 Implicit differentiation



We can differentiate any function that is written as a formula, and find a result in terms of a formula. However, sometimes the original problem can't be written in any nice way as a formula. For example, suppose we want to find dy/ dx in a case where the relationship be- tween x and y is given by the following equation:

$$y^7 + y = x^7 + x^2$$

There is no equivalent of the quadratic formula for seventh- order polynomials, so we have no way to solve for one variable in terms of the other in order to differentiate it. However, we can still find $dy/dx$ in terms of $x$ and $y$. Suppose we let $x$ grow to $x + dx$. Then for example the $x^2$ term will grow to $(x + dx)^2 = x + 2dx + dx^2$. The squared infinitesimal is negligible, so the increase in $x^2$ was re- ally just $2dx$, and we've really just computed the derivative of $x^2$with respect to $x$ and multiplied it by $dx$. In symbols,

$$\begin{aligned}d(x^2) &= \frac{d(x^2)}{dx}\cdot dx \\ &= 2xdx\end{aligned}$$

That is, the change in $x^2$ is $2x$ times the change in $x$. Doing this to both sides of the original equation, we have

$$\begin{aligned}d(y^7 + y) &= d(x^7 + x^2) \\ 7y^6dy + 1dy &= 7x^6dx + 2xdx \\ (7y^6 + 1)dy &= (7x^6 + 2x)dx \\ \frac{dy}{dx} &= \frac{7x^6 + 2x}{7y^6 + 1}\end{aligned}$$

This still doesn't give us a formula for the derivative in terms of $x$ alone, but it's not entirely use- less. For instance, if we're given a numerical value of $x$, we can al- ways use Newton's method to find $y$, and then evaluate the derivative.

## 5.3 Methods of integration Change of variable



Sometimes an unfamiliar-looking integral can be made into a familiar one by substituting a new variable for an old one. For example, we know how to integrate $1/x$ — the answer is $\text{In}\, x$ — but what about

$$\int \frac{dx}{2x+1}$$

Let $u = 2x + 1$. Differentiating both sides, we have $du = 2dx$, or $dx = du/2$,so

$$\begin{aligned}\int \frac{dx}{2x+1} &= \int \frac{du/2}{u} \\ &= \frac{1}{2}\text{In}\ u + c \\ &= \frac{1}{2}\text{In}\ (2x+1) + c\end{aligned}$$

This technique is known as a change of variable or a substitution. (Because the letter u is of- ten employed, you may also see it called $u$-substitution.)

In the case of a definite integral, we have to remember to change the limits of integration to reflect the new variable.

## Example

Evaluate $\int_3^4 dx/(2x+1)$.

As before, let $u = 2x + 1$.

$$\int_{x=3}^{x=4} \frac{dx}{2x+1} = \int_{u=7}^{u=9} \frac{du/2}{u}$$
$$= \frac{1}{2}\text{In } u \,|_{u=7}^{u=9}$$

Here the notation $|_{u=7}^{u=9}$ means to evaluate the function at 7 and 9, and sub- tract the former from the latter. The result is

$$\int_{x=3}^{x=4} \frac{dx}{2x+1} = \frac{1}{2}(\text{In } 9 - \text{In } 7)$$
$$= \frac{1}{2}\text{In}\frac{9}{7}$$

Sometimes, as in the next example, a clever substitution is the secret to doing a seemingly impossible integral.

## Example

Evaluate

$$\int \frac{e\sqrt{x}}{\sqrt{x}}dx$$

The only hope for reducing this to a form we can do is to let $u = \sqrt{x}$. Then $dx = d(u^2) = 2udu$, so

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int \frac{e^u}{u} \cdot 2udu$$
$$= 2\int e^u du$$
$$= 2e^u$$
$$= 2e^{\sqrt{x}}$$

Example (Page 105) really isn't so tricky, since there was only one logical choice for the substitution that had any hope of working. The following is a little more dastardly.

## Example

Evaluate

$$\int \frac{dx}{1+x^2}$$

The substitution that works is $x = \tan u$. First let's see what this does to the expression $1 + x^2$. The familiar identity

$$\sin^2 u + \cos^2 u = 1$$

when divided by $cos^2 u$, gives

$$\tan^2 u + 1 = \sec^2 u$$

so $1 + x^2$ becomes $\sec^2 u$. But differentiating both sides of $x = \tan u$ gives

$$\begin{aligned} dx &= d[\sin u(\cos u)^{-1}] \\ &= (\mathrm{dsin} u)(\cos u)^{-1} + (\sin u)d\left[(\cos u)^{-1}\right] \\ &= (1+\tan^2 u)du \\ &= \sec^2 u du \end{aligned}$$

so the integral becomes

$$\begin{aligned} \int \frac{dx}{1+x^2} &= \int \frac{\sec^2 u du}{\sec^2 u} \\ &= u + c \\ &= \tan^{-1} x + c \end{aligned}$$

```
Integrate(x) 1/(1+x^2)
ArcTan(x)
```

Another possible answer is that you can usually smell the possibility of this type of substitution, involving a trig function, when the thing to be integrated contains something reminiscent of the Pythagorean theorem, as suggested by Figure 5.2. The $1 + x^2$ looks like what you'd get if you had a right triangle with legs 1 and $x$, and were using the Pythagorean theorem to find its hypotenuse.

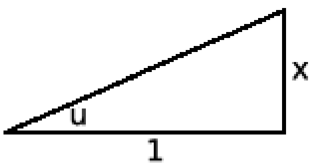


**Fig. 5.2: The substitution x = tan u.**

## Example

Evaluate $\int dx/\sqrt{1-x^2}$

The $\sqrt{1-x^2}$ looks like what you'd get if you had a right triangle with hypotenuse 1 and a leg of length $x$ , and were using the Pythagorean theorem to find the other leg, as in Figure 5.3. This motivates us to try the substitution $x = \cos\ u$, which gives $dx = -\sin\ udu$ and $\sqrt{1-x^2} = \sqrt{1-\cos^2 u} = \sin\ u$. The result is

$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{\text{-}\sin\ udu}{\sin\ u} \\ &= -u + c \\ &= -\cos^{-1} x\end{aligned}$$

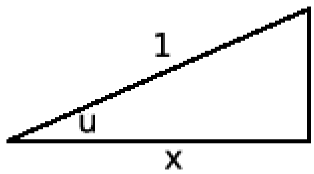


**Fig. 5.3: The substitution x = cos u.**

## 5.3.1 Integration by parts



Figure 5.4 shows a technique called integration by parts. If the integral $\int vdu$ is easier than the integral $\int udv$, then we can calculate the easier one, and then by simple geometry determine the one we wanted. Identifying the large rectangle that surrounds both shaded areas, and the small white rectangle on the lower left, we have

$$\int udv = (areaoflargerectangle) - (areaofsmallrectangle) \int vdu$$

In the case of an indefinite integral, we have a similar relationship de- rived from the product rule:

$$\begin{aligned}d(uv) &= udv + vdu \\ udv &= d(uv) - vdu\end{aligned}$$

Integrating both sides, we have the following relation.

## Integration by parts

$$\int u dv = uv - \int v du$$

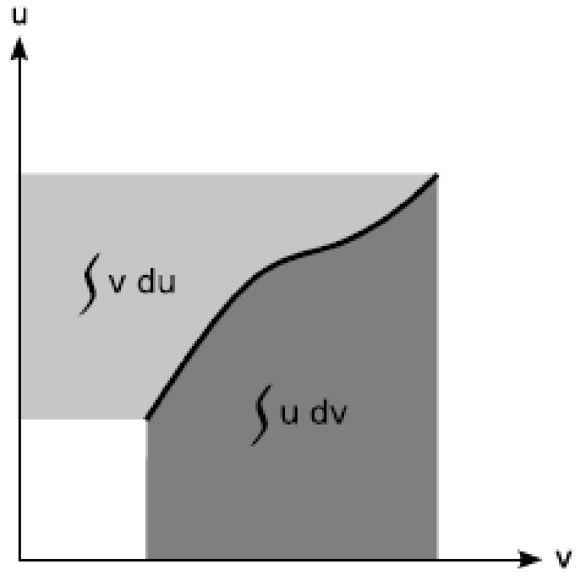


**Fig. 5.4: Integration by parts**

Since a definite integral can always be done by evaluating an indefinite integral at its upper and lower limits, one usually uses this form. Integrals don't usually come prepackaged in a form that makes it obvious that you should use integration by parts. What the equation for integration by parts tells us is that if we can split up the integrand into two factors, one of which (the $dv$) we know how to integrate, we have the option of changing the integral into a new form in which that factor becomes its integral, and the other factor becomes its derivative. If we choose the right way of splitting up the integrand into parts, the result can be a simplification.

## Example

Evaluate

$$\int x \cos x dx$$

There are two obvious possibilities for splitting up the integrand into factors,

$$udv = (x)(\cos x dx)$$

or

$$udv = (\cos x)(x dx)$$

The first one is the one that lets us make progress. If $u = x$, then $du = dx$, and if $dv = \cos x dx$, then integration gives $v = \sin x$.

$$\begin{aligned} \int x \cos x dx &= \int udv \\ &= uv - \int vdu \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x \end{aligned}$$

Of the two possibilities we considered for $u$ and $dv$ , the reason this one helped was that differentiating $x$ gave $dx$ , which was simpler, and integrating $\cos x dx$ gave $\sin x$ , which was no more complicated than before. The second possibility would have made things worse rather than better, because integrating $x dx$ would have given $x^2/2$, which would have been more complicated rather than less.

## Example

Evaluate $\int \text{In } x dx$

This one is a little tricky, because it isn't explicitly written as a product, and yet we can attack it using integration by parts. Let $u = \text{In } x$ and $dv = dx$

$$\begin{aligned} \int \text{In } x dx &= \int udv \\ &= uv - \int vdu \\ &= x\text{In}x - \int x \frac{dx}{x} \\ &= x\text{In}x - x \end{aligned}$$

## Example

Evaluate $\int x^2 e^x dx$

Integration by parts lets us split the integrand into two factors, integrate one, differentiate the other, and then do that integral. Integrating or differentiating $e^x$ does nothing. Integrating $x^2$ increases the exponent, which makes the problem look harder, whereas differentiating $x^2$ knocks the exponent down a step, which makes it look easier. Let $u = x^2$ and $dv = e^x dx$, so that $du = 2x dx$ and $v = e^x$. We then have

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx$$

Although we don't immediately know how to evaluate this new integral, we can subject it to the same type of inte- gration by parts, now with $u = x$ and $dv = e^x dx$. After the second integra- tion by parts, we have:

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2\left(x e^x - \int e^x dx\right) \\ &= x^2 e^x - 2(x e^x - e^x) \\ &= (x^2 - 2x + 2)e^x \end{aligned}$$

## 5.3.2 Partial fractions



Given a function like

$$\frac{-1}{x-1} + \frac{1}{x+1}$$

we can rewrite it over a common denominator like this:

$$\left(\frac{-1}{x-1}\right)\left(\frac{x+1}{x+1}\right) + \left(\frac{1}{x+1}\right)\left(\frac{x-1}{x-1}\right)$$

$$= \frac{-x-1+x-1}{(x-1)(x+1)}$$

$$= \frac{-2}{x^2-1}$$

But note that the original form is easily integrated to give

$$\int \left(\frac{-1}{x-1} + \frac{1}{x+1}\right) dx$$

$$= -\text{In }(x-1) + \text{In}(x+1) + c$$

While faced with the form $-2/(x^2-1)$, we wouldn't have known how to integrate it.

Note that the original function was of the form $(-1)/\cdots+(+1)/\cdots$ It's not a coincidence that the two constants on top, −1 and +1, are opposite in sign but equal in absolute value. To see why, consider the behavior of this function for large values of x. Looking at the form $-1/(x-1)+1/(x+1)$, we might naively guess that for a large value of $x$ such as 1000, it would come out to be somewhere on the order thousandths. But looking at the form $-2/(x^2-1)$, we would expect it to be way down in the millionths. This seeming paradox is resolved by noting that for large values of $x$, the two terms in the form $-1/(x-1)+1/(x+1)$ very nearly cancel. This cancellation could only have happened if the constants on top were opposites like plus and minus one.

The idea of the method of partial fractions is that if we want to do an integral of the form

$$\int \frac{dx}{P(x)}$$

where $P(x)$ is an nth order polynomial, we rewrite $1/P$ as

$$\frac{1}{P(x)}=\frac{A_1}{x-r_1}+\cdots\frac{A_n}{x-r_n}$$

where $r_1 \ldots r_n$ are the roots of the polynomial, i.e., the solutions of the equation $P(r)=0$. If the polynomial is second-order, you can find the roots $r_1$ and $r_2$ using the quadratic formula; I'll assume for the time being that they're real. For higher-order polynomials, there is no surefire, easy way of finding the roots by hand, and you'd be smart simply to use computer software to do it. In Yacas, you canfind the real roots of a polynomial like this:

```
FindRealRoots(x^4-5*x^3
-25*x^2+65*x+84)
f3.,7.,-4.,-1.g
```

(I assume it uses Newton's method to nd them.) The constants $A_i$ can then be determined by algebra, or by the following trick.

## 5.3.2.1 Numerical method



Suppose we evaluate $1/P(x)$ for a value of $x$ very close to one of the roots. In the example of the polynomial $x^4-5x^3-25x^2+65x+84$, let $r_1 \ldots r_4$ be the roots in the order in which they were re-turned by Yacas. Then $A_1$ can be found by evaluating $1/P(x)$ at $x=3.000001$:

```
P(x):=x^4-5*x^3-25*x^2
 +65*x+84
```

```
N(1/P(3.000001))
 -8928.5702094768
```

We know that for $x$ very close to 3, the expression

$$\frac{1}{P}=\frac{A_1}{x-3}+\frac{A_2}{x-7}+\frac{A_3}{x+4}+\frac{A_4}{x+1}$$

will be dominated by the $A_1$ term, so

$$-8930\approx\frac{A_1}{3.000001-3}$$
$$A_1\approx(-8930)(10^{-6})$$

By the same method we can find the other four constants:

```
dx:=.000001
N(1/P(7+dx),30)*dx
 0.2840908276e-2
N(1/P(-4+dx),30)*dx
 -0.4329006192e-2
N(1/P(-1+dx),30)*dx
 0.1041666664e-1
```

(The `N( ,30)` construct is to tell Yacas to do a numerical calculation rather than an exact symbolic one, and to use 30 digits of precision, in order to avoid problems with rounding errors.) Thus,

$$\frac{1}{P}=\frac{-8.93\times 10^{-3}}{-3}+\frac{2.84\times 10^{-3}}{x-7}-\frac{4.33\times 10^{-3}}{x+4}+\frac{1.04\times 10^{-2}}{x+1}$$

The desired integral is

$$\int\frac{dx}{P(x)}=-8.93\times 10^{-3}\text{In}(x-3)+2.84\times 10^{-3}\text{In}(x-7)-4.33\times 10^{-3}\text{In}(x+4)+1.04\times 10^{-2}\text{In}(x+1)+c$$

As in the simpler example I started off with, where $P$ was second or- der and we got $A_1=-A_2$, in this $n=4$ example we expect that $A_1+A_2+A_3+A_4=0$, for otherwise the large-$x$ behavior of the partial-fraction form would be $1/x$ rather than

$1/x^4$ . This is a useful way of checking the result: $-8.93 + 2.84 - 4.33 + 10.4 = -.02 \approx 0.$

### 5.3.2.2 Complications



There are two possible complications:

First, the same factor may occur more than once, as in $x^3 = 5x^2 + 7x - 3 = (x-1)(x-1)(x-3)$. In this example, we have to look for an answer of the form $A/(x-1) + B/(x-1)^2 + C/(x-3)$, the solution being $-.25/(x-1) - .5/(x-1)^2 + .25/(x-3)$.

Second, the roots may be complex. This is no show-stopper if you're using computer software that handles complex numbers gracefully. (You can choose a c that makes the result real.) In fact, as discussed in Partial fractions revisited (Page 151), some beautiful things can happen with complex roots. But as an alternative, any polynomial with real coefficients can be factored into linear and quadratic factors with real coefficients. For each quadratic factor $Q(x)$, we then have a partial fraction of the form $(A + Bx)/Q(x)$, where $A$ and $B$ can be determined by algebra. In Yacas, this can be done using the `Apart` function.

**Example**

Evaluate the integral

$$\int \frac{dx}{(x^4 - 8x^3 + 8x^2 - 8x + 7)}$$

using the method of partial fractions.

```
FindRealRoots(x^4-8*x^3
+8*x^2-8*x+7)
{f1.,7.}
```

```
Apart(1/(x^4-8*x^3
 +8*x^2-8*x+7))
 ((2*x)/25+3/50)/(x^2+1)
 +1/(300*(x-7))
 +(-1)/(12*(x-1))
```

We can now rewrite the integral like this:

$$\frac{2}{25}\int\frac{xdx}{x^2+1}$$
$$+\frac{3}{50}\int\frac{dx}{x^2+1}$$
$$+\frac{1}{300}\int\frac{dx}{x-7}$$
$$-\frac{1}{12}\int\frac{dx}{x-1}$$

which we can evaluate as follows:

$$\frac{1}{25}\text{In}(x^2+1)$$

$$+\frac{3}{50}\tan^{-1}x$$
$$+\frac{1}{300}\text{In}(x-7)$$
$$-\frac{1}{12}\text{In}(x-1)$$
$$+c$$

In fact, Yacas should be able to do the whole integral for us from scratch, but it's best to understand how these things work under the hood, and to avoid being completely dependent on one particular piece of software. As an illustration of this gem of wisdom, I found that when I tried to make Yacas evaluate the integral in one gulp, it choked because the calculation became too complicated! Because I understood the ideas behind the procedure, I was still able to get a result through a mixture of computer calculations and working it by hand. Some- one who didn't have the knowledge of the technique might have tried the integral using the software, seen it fail, and concluded, incorrectly, that the integral was one that simply couldn't be done. A computer is no substitute for understanding.

#### 5.3.2.3 Residue method



In Partial fractions (Page 110) I introduced the trick of carrying out the method of partial fractions by evaluating $1/P(x)$ numerically at $x=r_i+\epsilon$, near where $1/P$ blows up. Sometimes we would like to have an exact result rather than a numerical approximation. We can accomplish this by using an infinitesimal number dx rather than a small but finite $\epsilon$. For simplicity, let's assume that all of the $n$ roots $r_i$ are distinct, and that $P$'s highest-order term is $x^n$. We can then write $P$ as the product $P(x)=(x-r_1)(x-r_2)...(x-r_n)$. For products like this, there is a notation $\prod$ (capital Greek letter "pi") that works like $\sum$ does for sums:

$$P(x)=\prod_{i=1}^{n}(x-r_i)$$

It's not necessary that the roots be real, but for now we assume that they are. We want to find the coefficients $A_i$ such that

$$\frac{1}{P(x)} = \sum \frac{A_i}{x - r_i}$$

We then have

$$\begin{aligned} &\frac{1}{P(r_i + dx)} \\ &= \frac{1}{dx \prod_{j \neq i}(r_i - r_j + dx)} \\ &= \frac{1}{dx \prod_{j \neq i}(r_i - r_j)} + \cdots \\ &= \frac{A_i}{dx} + \cdots \end{aligned}$$

where ... represents finite terms that are negligible compared to the infinite ones. Multiplying on both sides by $dx$, we have

$$\frac{1}{P'(r_i)} + \cdots = A_i + \cdots$$

where the ... now stand for infinitesimals which must in fact cancel out, since both $A_i$i and $1/P'$ are real numbers.

## Example

The partial-fraction decomposition of the function

$$\frac{1}{x^4 - 5x^3 - 25x^2 + 65x + 84}$$

was found numerically on Partial fractions (Page 110). The coefficient of the $1/(x-3)$ term was found numerically to be $A_1 \approx -8.930 \times 10^{-3}$. Determine it exactly using the residue method.

Differentiation gives $P'(x) = 4x^3 - 15x^2 - 50x + 65$. We then have $A_1 = 1/P'(3) = -1/112$.

## 5.3.3 Integrals that can't be done



Integral calculus was invented in the age of powdered wigs and harpsichords, so the original emphasis was on expressing integrals in a form that would allow numbers to be plugged in for easy numerical evaluation by scribbling on scraps of parchment with a quill pen. This was an era when you might have to travel to a large city to get access to a table of logarithms.

In this computationally impoverished environment, one always wanted to get answers in what's known *as closed form* and in terms of *elementary functions*.

A closed form expression means one written using a finite number of operations, as opposed to something like the geometric series $1 + x + x^2 + x^3 + \ldots$, which goes on forever.

Elementary functions are usually taken to be addition, subtraction, multiplication, division, logs, and exponentials, as well as other functions derivable from these. For ex- ample, a cube root is allowed, since $3\sqrt{x} = e^{(1/3)\text{In x}}$ , and so are trig functions and their inverses, since, as we will see in chapter 8, they can be expressed in terms of logs and exponentials.

In theory, "closed form" doesn't mean anything unless we state the elementary functions that are al- lowed. In practice, when people refer to closed form, they usually have in mind the particular set of elementary functions described above.

A traditional freshman calculus course spends such a vast amount of time teaching you how to do integrals in closed form that it may be easy to miss the fact that this is impossible for the vast majority of integrands that you might randomly write down. Here are some examples of impossible integrals:

$$\int e^{-x^3} dx$$

$$\int x^x dx$$

$$\int \frac{\sin\, x}{x} dx$$

$$\int e^x \tan\, x dx$$

The first of these is a form that is extremely important in statistics (it describes the area under the standard "bell curve"), so you can see that impossible integrals aren't just obscure things that don't pop up in real life.

People who are proficient at doing integrals in closed form generally seem to work by a process of pat- tern matching. They recognize certain integrals as being of a form that can't be done, so they know not to try.

## Example

Students! Stand at attention! You will now evaluate $\int e^{-x^2+7x} dx$ in closed form.

No sir, I can't do that. By a change of variables of the form $u = x + c$, where $c$ is a constant, we could clearly put this into the form $\int e^{-x^2} dx$ know is impossible.

Sometimes an integral such as $\int e^{-x^2} dx$ is important enough that we want to give it a name, tabulate it, and write computer subroutines that can evaluate it numerically.

For example, statisticians define the “error function” $\text{erf }(x) = (2/\sqrt{\pi}) \int e^{-x^2} dx$ Sometimes if you're not sure whether an integral can be done in closed form, you can put it into computer software, which will tell you that it reduces to one of these functions. You then know that it can't be done in closed form. For example, if you ask the popular web site integrals.com to do $\int e^{-x^2+7x} dx$, it spits back $(1/2)e^{49/4}\sqrt{\pi}\ \text{erf}(x-7/2)$. This tells you both that you shouldn't be wasting your time trying to do the integral in closed form and that if you need to evaluate it numerically, you can do that using the erf function.

As shown in the following example, just because an indefinite integral can't be done, that doesn't mean that we can never do a related definite integral.

## Example

Evaluate $\int_0^{\pi/2} e^{-\tan^2 x}(\tan^2 x + 1)dx$.

The obvious substitution to try is $u = \tan\ x$, and this reduces the integrand to $e^{-x^2}$. This proves that the corre- sponding indefinite integral is impossible to express in closed form. How- ever, the definite integral can be expressed in closed form; it turns out to be $\sqrt{\pi}/2$. The trick for proving this is given in Example 99 (Page 163).

Sometimes computer software can't say anything about a particular integral at all. That doesn't mean that the integral can't be done. Computers are stupid, and they may try brute-force techniques that fail because the computer runs out of memory or CPU time For example, the integral $\int dx/(x^1 0000 - 1)$ (Problem 8.5 (Page 154)) can be done in closed form using the techniques of chapter 8, and it's not too hard for a proficient human to figure out how to attack it, but every computer program I've tried it on has failed silently.

# 5.4 Problems

## 5.4.1 Problem 5.1



Graph the function $y = e^x - 7x$ and get an approximate idea of where any of its zeroes are (i.e., for what values of $x$ we have $y(x) = 0$). Use Newton's method to find the zeroes to three significant figures of precision.

## 5.4.2 Problem 5.2



The relationship between $x$ and $y$ is given by $xy = \sin\ y + x^2y^2$. (a) Use Newton's method to find the nonzero solution for $y$ when $x = 3$. Answer: $y = 0.2231$ (b) Find

$dy/dx$ in terms of $x$ and $y$, and evaluate the derivative at the point on the curve you found in part a. Answer: $dy/dx = -0.0379$ Based on an example by Craig B. Watkins.

### 5.4.3 Problem 5.3



Suppose you want to evaluate $\int \frac{dx}{1+\sin 2x}$ and you've found $\int \frac{dx}{1+\sin x} = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$ in a table of integrals. Use a change of variable to find the answer to the original problem.

### 5.4.4 Problem 5.4



Evaluate

$$\int \frac{\sin x dx}{1+\cos x}$$

### 5.4.5 Problem 5.5



Evaluate

$$\int \frac{\sin x dx}{1+\cos^2 x}$$

### 5.4.6 Problem 5.6



Evaluate

$$\int x\sqrt{a-x}dx$$

### 5.4.7 Problem 5.7



Evaluate

$$\int \sqrt{x^4+bx^2}dx$$

where $b$ is $a$ constant.

### 5.4.8 Problem 5.8



Evaluate

$$\int xe^{-x^2}\,dx$$

### 5.4.9 Problem 5.9



Evaluate

$$\int xe^{x}\,dx$$

### 5.4.10 Problem 5.10



Use integration by parts to evaluate the following integrals.

$$\int \sin^{-1} x dx$$

$$\int \cos^{-1} x dx$$

$$\int \tan^{-1} x dx$$

### 5.4.11 Problem 5.11



Evaluate

$$\int x^2 \sin\, x dx$$

Hint: Use integration by parts more than once.

### 5.4.12 Problem 5.12



Evaluate

$$\int \frac{dx}{x^2 - x - 6}$$

## 5.4.13 Problem 5.13



Evaluate

$$\int \frac{dx}{x^3 + 3x^2 - 4}$$

## 5.4.14 Problem 5.14



Evaluate

$$\int \frac{dx}{x^3 - x^2 + 4x - 4}$$

## 5.4.15 Problem 5.15



Apply integration by parts twice to

$$\int e^{-x} \cos x dx$$

examine what happens, and manipulate the result in order to solve the original integral. (An approach that doesn't rely on tricks is given in Example 91 (Page 150))

## 5.4.16 Problem 5.16



1. Plan, but do not actually carry out the steps that would be required in order to generalize the result of Example (Page 110) in order to evaluate

$$\int x^a b^{-x} dx$$

where $a$ and $b$ are constants. Which is easier, the generalization from 2 to $a$, or the one from $e$ to $b$? Do we need to introduce any restrictions on $a$ or $b$?

Solutions for chapter 5 (Page 211)

## 5.4.17 Problem 5.17



The integral $\int e^{-x^2} dx$ can't be done in closed form. Knowing this, use a change of variable to write down a different integral that also can't be done in closed form.

### 5.4.18 Problem 5.18



Consider the integral

$$\int e^{x^p}\,dx$$

where $p$ is a constant. There is an obvious substitution. If this is to result in an integral that can be evaluated in closed form by a series of integrations by parts, what are the possible values of $p$? Don't actually complete the integral; just determine what values of $p$ will work.

Solutions for chapter 5 (Page 211)

### 5.4.19 Problem 5.19



Evaluate the hundredth derivative of the function $(x^2+1)/(x^3-x)$using paper and pencil. [Vladimir Arnol'd] .

Solutions for chapter 5 (Page 211)

# Chapter 6 Improper integrals

## 6.1 Integrating a function that blows up



When we integrate a function that blows up to infinity at some point in the interval we're integrating, the result may be either finite or infinite.

### Example 75

Integrate the function $y = 1\sqrt{x}$ from x = 0 to x = 1.
The function blows up to infinity at one end of the region of integration, but let's just try evaluating it, and see what happens.

$$\int_0^1 x^{-1/2} dx = 2x^{1/2} \mid_0^1$$
$$= 2$$

The result turns out to be finite. Intuitively, the reason for this is that the spike at *x*= 0 is very skinny, and gets skinny fast as we go higher and higher up.

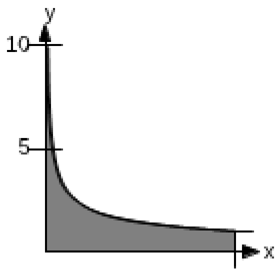


**Fig. 6.1:**

a The integral $\int_0^1 dx/\sqrt{x}$ is finite.

**Fig. 6.2:**

### Example 76

Integrate the function $y = 1/x^2$ from *x*= 0 to *x*= 1.

$$\int_0^1 x^{-2} dx = x^{-1} \mid_0^1$$
$$= -1 + \frac{1}{0}$$

Division by zero is undefined, so the result is undefined.
Another way of putting it, using the hyperreal number system, is that if we were to integrate from $\varepsilon$ to 1, where was an infinitesimal number, then the result would be − 1 + 1=$\varepsilon$ , which is infinite. The smaller we make , the bigger the infinite result we get out.
Intuitively, the reason that this integral comes out infinite is that the spike at *x*= 0 is fat, and doesn't get skinny fast enough.

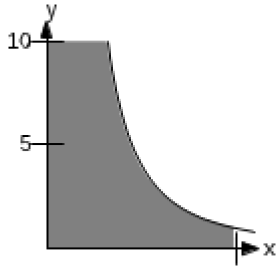


**Fig. 6.3:**

b / The integral $\int_0^1 dx/x^2$ is infinite.

These two examples were examples of improper integrals.

## 6.2 Limits of integration at infinity



Another type of improper integral is one in which one of the limits of integration is infinite. The notation

$$\int_a^\infty f(x)dx$$

means the limit of $\int_a^H f(x)dx$, where *H* is made to grow bigger and bigger. Alternatively, we can think of it as an integral in which the top end of the interval of integration is an infinite hyper real number. A similar interpretation applies when the lower limit is $-\infty$, or when both limits are infinite.

# Example 77

Evaluate

$$\int_1^\infty x^{-2}dx$$

$$\int_1^H x^{-2}dx = -x^{-1} \Big|_1^H$$
$$= -\frac{1}{H} + 1$$

As H gets bigger and bigger, the result gets closer and closer to 1, so the result of the improper integral is 1.
Note that this is the same graph as in example 75, but with the *x*and *y* axes interchanged; this shows that the two different types of improper integrals really aren't so different.

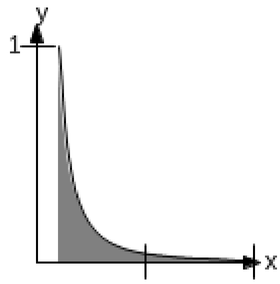


**Fig. 6.4:**

c / The integral $\int_1^\infty dx/x^2$ is finite.

# Example 78

Newton's law of gravity states that the gravitational force between two objects is given by $F = Gm_1m_2/r^2$, where *G* is a constant, $m_1$ and $m_2$ are the objects' masses, and *r* is the center-to-center distance between them. Compute the work that must be done to take an object from the earth's surface, at *r* = *a*, and remove it to *r*= ∞.

$$W = \int_a^\infty \frac{Gm_1m_2}{r^2}dr$$
$$= Gm_1m_2 \int_a^\infty r^{-2}dr$$
$$= -Gm_1m_2r^{-1} \Big|_a^\infty$$
$$= \frac{Gm_1m_2}{a}$$

The answer is inversely proportional to a. In other words, if we were able to start from higher up, less work would have to be done.

## 6.3 Problems

### 6.3.1 Problem 6.1



Integrate

$$\int_0^\infty e^{-x} dx,$$

or show that it diverges.

### 6.3.2 Problem 6.2



Integrate

$$\int_1^\infty \frac{dx}{x},$$

or show that it diverges.

### 6.3.3 Problem 6.3



Integrate

$$\int_0^1 \frac{dx}{x},$$

or show that it diverges.

### 6.3.4 Problem 6.4



Integrate

$$\int_0^\infty x^2 2^{-x} dx,$$

or show that it diverges.

Solutions for chapter 6 (Page 212)

### 6.3.5 Problem 6.5



Integrate

$$\int_0^\infty e^{-x} cos x dx$$

or show that it diverges. (Problem 5.15 (Page 120) suggests a trick for doing the indefinite integral.)

### 6.3.6 Problem 6.6



Prove that

$$\int_0^\infty e^{-e^x}\,dx,$$

converges, but don't evaluate it.

### 6.3.7 Problem 6.7



(a) Verify that the probability distribution d$P$ / d$x$ given in example 60 on page 80 is properly normalized.
(b) Find the average value of $x$, or show that it diverges.
(c) Find the standard deviation of $x$, or show that it diverges.

### 6.3.8 Problem 6.8



Prove

$$\int_0^\infty e^{-x} x^n dx = n!$$

# Chapter 7 Sequences and Series

## 7.1 Infinite sequences



Consider an infinite sequence of numbers like 1/2, 2/3, 3/4, 4/5,. . . We want to define this as approaching 1, or "converging to 1." The way to do this is to make a function *f*(*n*), which is only well defined for integer values of *n*. Then *f*(1) = 1/2, *f*(2) = 2/3, and in general *f*(*n*) = *n*/(*n*+ 1). With just a little tinkering, our definitions of limits can be applied to this type of function (see Problem 7.1 (Page 138) ).

## 7.2 Infinite series



A related question is how to rigorously define the sum of infinitely many numbers, which is referred to as an infinite *series*. An example is the geometric series 1 + *x*+ $x^2$ + $x^3$ + ...= 1/(1 −*x*), which we used casually on page 29. The general concept of an infinite series goes back to ancient Greek mathematics. Various supposed para- doxes about infinite series, such as Zeno's paradox, were exhibited, influencing Euclid to sidestep the is- sue in his *Elements*, where in Book IX, Proposition 35 he provides only an expression $(1-x^n)/(1-x)$ for the *n*th partial sum of the geometric series. The case where *n*gets so big that $x^n$ becomes negligible is left to the reader's imagination, as in one of those scenes in a romance novel that ends with something like "...and she surrendered..." For those with modern training, the idea is that an infinite sum like 1 + 1 + 1 + ...would clearly give an infinite result, but this is only because the terms are all staying the same size. If the terms get smaller and smaller, and get smaller fast enough, then the result can be finite. For example, consider the geometric series in the case where *x*= 1/2, for which we expect the result 1/(1 −1/2) = 2. We have

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots,$$

which at the successive steps of addition equals $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16}, \ldots.$ We're getting closer and closer to 2, cutting the distance in half at each step. Clearly we can get as close as we like to 2, if we're willing to add enough terms.

Note that we ended up wanting to talk about the partial sums of the series. This is the right way to get a rigorous definition of the convergence of series in general. In the case of the geometric series, for ex- ample, we can define a sequence of the partial sums 1, 1 + *x*, 1 + *x*+ $x^2$. . We can then define convergence and limits of series in terms of convergence and limits of the partial

It's instructive to see what happens to the geometric series with *x*= 0.1. The geometric series becomes

1 + 0.1 + 0.01 + 0.001+ ... .

The partial sums are 1, 1.1, 1.11, 1.111, . We can see vividly here that adding another term will only affect the result in a certain decimal place, without affecting any of the earlier ones. For instance, if we needed a result that was valid to three digits past the decimal place, we could stop at 1.111, being assured that we had attained a good enough approximation. If we wanted an exact result, we could also observe that multiplying the result by 9 would give 9.999 ..., which is the same as 10, so the result must be 10/9, which is in agreement with 1/(1 −1/10) = 10/9.

One thing to watch out for with infinite series is that the axioms of the real number system only talk about finite sums, so it's easy to get wrong results by attempting to apply them to infinite ones (see Problem 7.2 (Page 139)).

## 7.3 Tests for convergence



There are many different tests that can be used to determine whether a sequence or series converges. I'll briefly state three of the most useful, with sketches of their proofs.

*Bounded and increasing sequences:* A sequence that always increases, but never surpasses a certain value, converges.

This amounts to a restatement of the completeness axiom forThe intermediate value theorem (Page 183), and is therefore to be interpreted not so much as a statement about sequences but as one about the real number system. In particular, it fails if interpreted as a statement about sequences confined entirely to the rational number system, as we can see from the sequence 1, 1.4, 1.41,1.414, . . . consisting of the successive decimal approximations to $\sqrt{2}$, which does not converge to any rational-number value.

### Example 79

Prove that the geometric series 1 +1/2 + 1/4 +...converges.
The sequence of partial sums is in- creasing, since each term is positive. Each term closes half of the remaining gap separating the previous partial sum from 2, so the sum never surpasses 2. Since the partial sums are increasing and bounded, they converge to a limit.
Once we know that a particular series converges, we can also easily infer the convergence of other series whose terms get smaller faster.
For example, we can be certain that if the geometric series con- verges, so does the series

$$\frac{1}{1} + \frac{1}{1X2} + \frac{1}{1X2X3} + \ldots,$$

whose terms get smaller faster than any base raised to the power *n*.

*Alternating series with terms approaching zero:* If the terms of a series alternate in sign and approach zero, then the series converges.

Sketch of a proof: The even partial sums form an increasing sequence, the odd sums a decreasing one. Neither of these sequences of partial sums can be unbounded, since the difference between partial sums *n*and *n*+ 1 would then have to be unbounded, but this difference is simply the *n*th term, and the terms approach zero. Since the even partial sums are increasing and bounded, they converge to a limit, and similarly for the odd ones. The two limits must be equal, since the terms approach zero.

## Example 80

Prove that the series 1 −1/2 + 1/3 −1/4 + ...converges.

Its convergence follows because it is an alternating series with decreasing terms. The sum turns out to be ln 2, although the convergence of the series is so slow that an extremely large number of terms is required in order to obtain a decent approximation,

*The integral test:* If the terms of a series *an*are positive and decreasing, and *f*(*x*) is a positive and decreasing function on the real number line such that *f*(*n*) = $a_n$, then the sum of $a_n$ from *n*= 1 to ∞ converges if and only if $\int_1^\infty f(x)dx$ dx does.

Sketch of proof: Since the theorem is supposed to hold for both convergence and divergence, and is also an "if and only if," there are actually four cases to prove, of which we pick the representative one where the integral is known to converge and we want to prove convergence of the corresponding sum. The sum and the integral can be interpreted as the areas under two graphs: one like a smooth ramp and one like a staircase. Sliding the staircase half a unit to the left, it lies entirely underneath the ramp, and therefore the area under it is also finite.

## Example 81

Prove that the series 1+1/2+1/3+...diverges.

The integral of 1/*x*is ln *x*, which diverges as *x* approaches infinity, so the series diverges as well.

*Theratio test:* If the limit *R*= lim*n*→∞|$a_n$+1 /$a_n$|exists, then the sum of *an* converges if *R*<1 and diverges if *R*>1.

The proof can be obtained by comparing with a geometric series.

## Example 82

Prove that the series 1+1/22 +1/33 +...converges.

*R* is easily proved to be 0, so the sum converges by the ratio test.

At this point it will seem like a mystery how anyone could have proved the exact results claimed for some of the "special" series, such as 1 -1/2 + 1/3 -1/4 +...= ln 2. Problems like these are not the main focus of the chapter, and in fact there is no well- defined toolbox of techniques that will allow any such "nice" series to be evaluated exactly. Even a relatively innocent-looking example like $1^{-2}+2^{-2}+3^{-2}+\ldots$defeated some of the best mathematicians of Europe for years (see problem 16, p. 116). It is currently unknown whether some apparently simple

series such as $\sum_{n=1}^{\infty} 1/(n^3 sin^2 n)$ converge [1].

# 7.4 Taylor series



If you calculate $e^{0.1}$ on your calculator, you'll find that it's very close to 1.1. This is because the tangent line at *x*= 0 on the graph of $e^x$ has a slope of 1 ( $de^x/dx = e^x = 1$ at *x*= 0), and the tangent line is a good approximation to the exponential curve as long as we don't get too far away from the point of tangency.

How big is the error? The actual value of $e^{0.1}$ is 1.10517091807565 ..., which differs from 1.1 by about 0.005. If we go farther from the point of tangency, the approximation gets worse. At *x*= 0.2, the error

1. 1Alekseyev, \On convergence of the Flint Hills series," arxiv.org/abs/1104. 5100v1

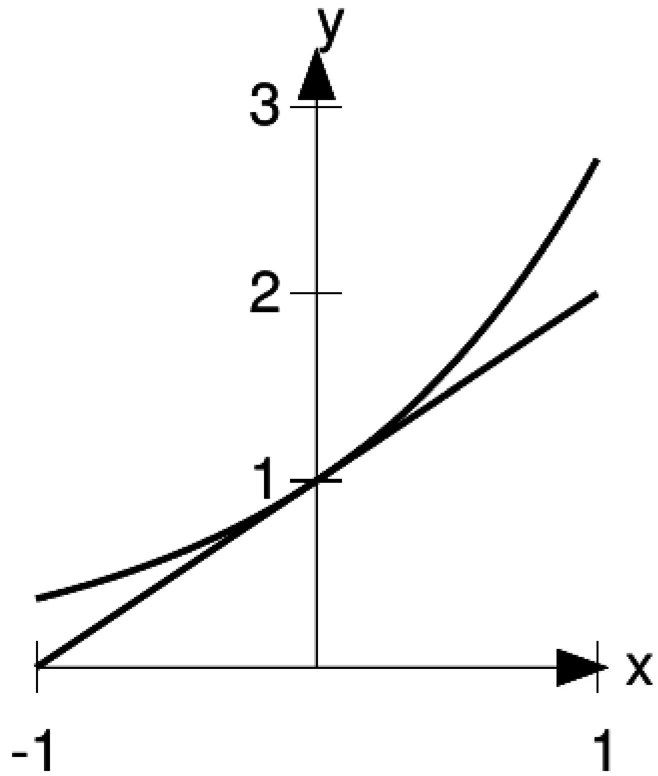


**Fig. 7.1:**

a / The function $e^x$ , and the tangent line at x = 0.

is about 0.021, which is about four times bigger. In other words, doubling x seems to roughly quadruple the error, so the error is proportional to $x^2$; it seems to be about $x^2$/2. Well, if we want a handy-dandy, super-accurate estimate of $e^x$ for small values of *x*, why not just account for this error. Our new and improved estimate is

$$e^x \approx 1 + x + \frac{1}{2}x^2$$

for small values of *x*.

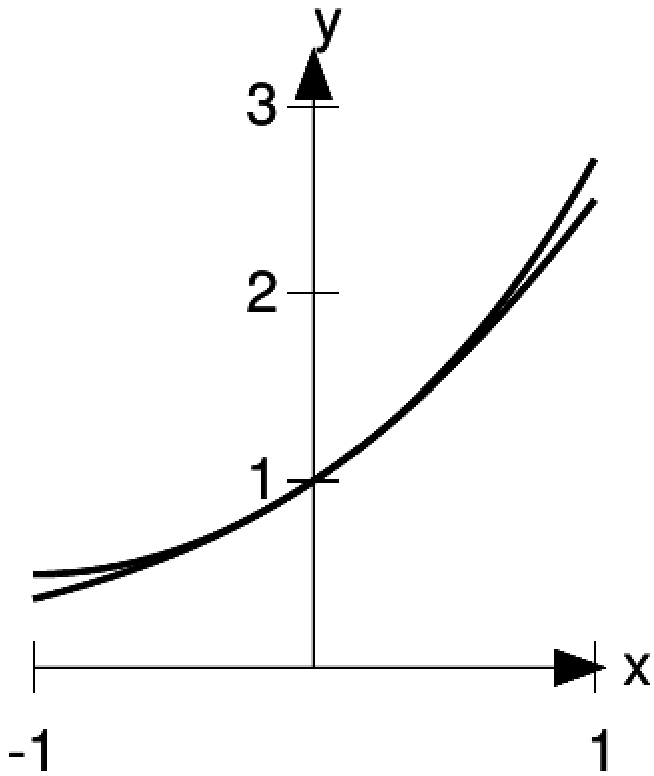


**Fig. 7.2:**

b / The function $e^x$ , and the approximation 1 + x + $x^2$ /2.

Figure b shows that the approximation is now extremely good for sufficiently small values of *x*. The difference is that whereas 1 + *x* matched both the y-intercept and the slope of the curve, 1 + *x*+ $x^2$ /2 matches the curvature as well. Recall that the second derivative is a measure of curvature. The second derivatives of the function and its approximation are

$$\frac{d}{dx}e^x = 1$$

$$\frac{d}{dx}(1 + x + \frac{1}{2}x^2) = 1$$

We can do even better. Suppose

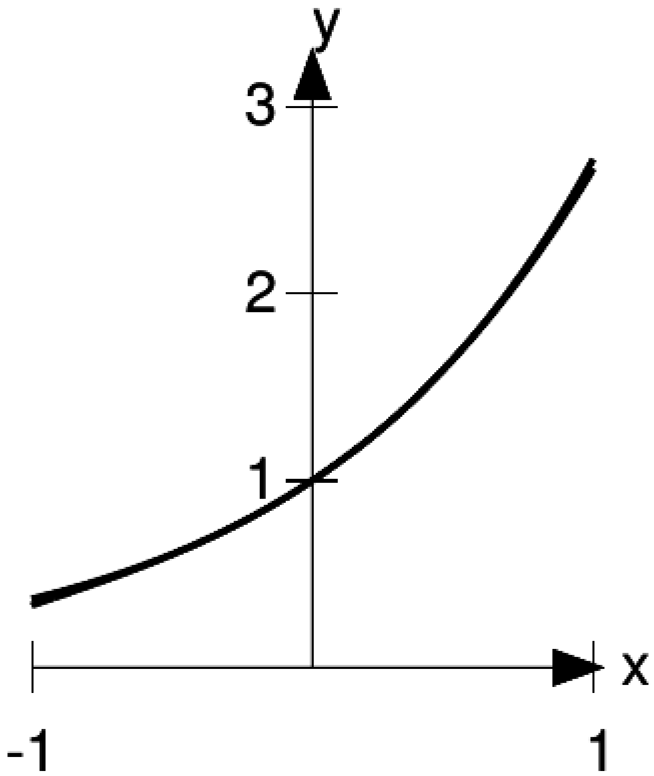


**Fig. 7.3:**

c / The function $e^x$, and the approximation $1 + x + x^2/2 + x^3/6$

we want to match the third derivatives. All the derivatives of ex, evaluated at x = 0, are 1, so we just need to add on a term proportional to x3 whose third derivative is one. Taking the rst derivative will bring down a factor of 3 in front, and taking and the second derivative will give a 2, so to cancel these out we need the third order term to be (1=2)(1=3):

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3$$

Figure c shows the result. For a significant range of *x*values close to zero, the approximation is now so good that we can’t even see the difference between the two functions on the graph.

On the other hand, figure d shows that the cubic approximation for somewhat larger negative and positive values of *x*is poor — worse, in fact, than the linear approximation, or even the constant approximation $e^x$ = 1. This is to be expected, because any polynomial will blow up to either positive or negative infinity as *x* approaches negative infinity, whereas the function $e^x$is supposed to get very close to zero for large negative *x*. The idea here is that derivatives are *local* things: they only measure the properties of a function very close to the point at which they’re evaluated, and they don’t necessarily tell us anything about points far away.

It's a remarkable fact, then, that by taking enough terms in a polynomial approximation, we can always get as good an approximation to ex as necessary | it's just that a large number of terms may be required for large values of x. In other words, the infnite series

$$1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3 + \ldots$$

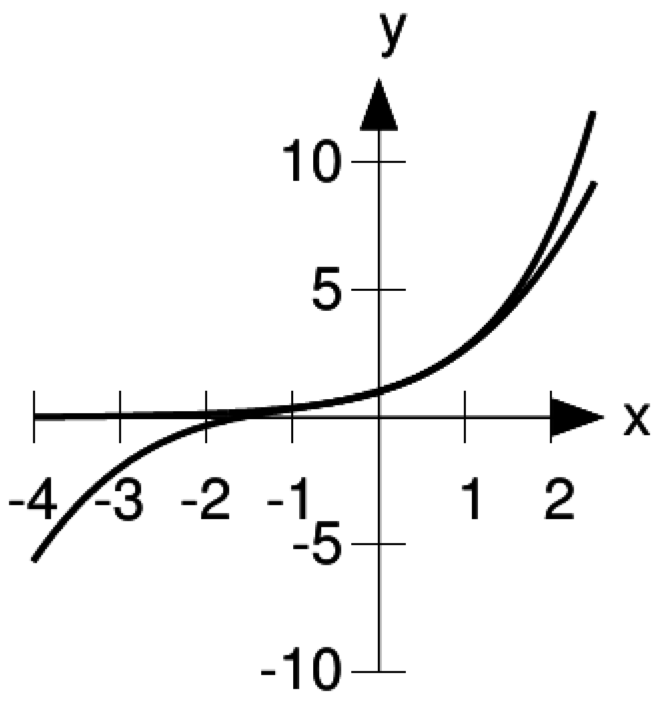


**Fig. 7.4:**

d / The function ex , and the approximation $1+x+x^2/2+x^3/6$, on a wider scale.

always gives exactly $e^x$. But what is the pattern here that would allows us to gure out, say, the fourth-order and fth-order terms that were swept under the rug with the symbol \. . . "? Let's do the fth-order term as an example. The point of adding in a fth-order term is to make the fth derivative of the approximation equal to the fth derivative of $e^x$, which is 1. The rst, second, . . . derivatives of x5 are

$$\frac{d}{dx}x^5 = 5x^4$$
$$\frac{d^2}{dx^2}x^5 = 5\cdot 4x^3$$
$$\frac{d^2}{dx^3}x^5 = 5\cdot 3x^2$$
$$\frac{d^4}{dx^4}x^5 = 5\cdot 4\cdot 3\cdot 2x$$
$$\frac{d^5}{dx^5}x^5 = 5\cdot 4\cdot 3\cdot 2\cdot 1$$

The notation for a product like 1 ·2 ·...·*n* is *n*!, read “*n* factorial.” So to get a term for our polynomial whose fifth derivative is 1, we need $x^5$ /5!. The result for the infinite series is

$$e^x = \sum_{\infty}^{n=0} \frac{x^n}{n}$$

where the special case of 0! = 1 is assumed. [2] This infinite series is called the *Taylor series* for $e^x$, evaluated around *x*= 0, and it’s true, although I haven’t proved it, that this particular Taylor series always converges to $e^x$, no matter how far *x* is from zero.

2. This makes sense, because, for exam- ple, 4!=5!/5, 3!=4!/4, etc., so we should have 0!=1!/1.

In general, the Taylor series around $x$= 0 for a function $y$ is

$$T_0(x) = \sum_{\infty}^{n=0} a_n x^n$$

where the condition for equality of the nth order derivative is

$$a_n = \frac{1}{n!}\frac{d^n y}{dx^n} \mid x = 0$$

Here the notation $\mid x = 0$ means that the derivative is to be evaluated at $x$= 0.

A Taylor series can be used to approximate other functions besides $e^x$, and when you ask your calculator to evaluate a function such as a sine or a cosine, it may actually be using a Taylor series to do it. Taylor series are also the method Inf uses to calculate most expressions involving infinitesimals. In example 13 on page 29, we saw that when Inf was asked to calculate 1/(1 −$d$), where $d$ was infinitesimal, the result was the geometric series:

: 1/(1-d)

*1+d+d^2+d^3+d^4*

These are also the the first five terms of the Taylor series for the function $y$= 1/(1 −$x$), evaluated around $x$= 0. That is, the geo- metric series 1 + $x$+ $x^2$ + $x^3$ + ... is really just one special example of a Taylor series, as demonstrated in the following example.

# Example 83

Find the Taylor series of $y$= 1/(1 −$x$) around $x$= 0.
Rewriting the function as $y = (1 - x)^{-1}$ and applying the chain rule, we have

$$y\,|_{x=0} = 1$$
$$\frac{dy}{dx}\,|_{x=0} = (1-x)^{-2}\big|_{x=0} = 1$$
$$\frac{d^2y}{dx^2}\,|_{x=0} = 2\,(1-x)^{-3}\big|_{x=0} = 2$$
$$\frac{d^3y}{dx^3}\,|_{x=0} = 2\cdot 3\,(1-x)^{-4}\big|_{x=0} = 2\cdot 3$$

The pattern is that the nth derivative is $n$!. The Taylor series therefore has $a_n$= $n$!/$n$! = 1:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

If you flip back to Tests for convergence (Page 128) and compare the rate of convergence of the geometric series for $x$ = 0.1 and 0.5, you'll see that the sum converged much more quickly for $x$= 0.1 than for $x$= 0.5. In general, we expect that any Taylor series will converge more quickly when xis smaller. Now consider what happens at $x$= 1. The series is now 1 + 1 + 1 + ..., which gives an infinite result, and we shouldn't have expected any better behavior, since attempting to evaluate 1/(1 -*x*) at *x* =1 gives division by zero. For *x*>1, the results become nonsense. For example, 1/(1 -2) = -1, which is finite, but the geometric series gives 1 + 2 + 4 + ..., which is infinite.

In general, every function's Taylor series around *x*= 0 converges for all values of *x*in the range defined by |*x*|<*r*, where *r*is some number, known as the radius of convergence. Also, if the function is defined by putting together other functions that are well behaved (in the sense of converging to their own Taylor series in the relevant region), then the Taylor series will not only converge but converge to the *correct*value. For the function *ex*, the radius happen to be infinite, whereas for 1/(1 -*x*) it equals 1. The following example shows a worst-case scenario.

# Example 84

The function $y = e^{-1/x^2}$, shown in figure e

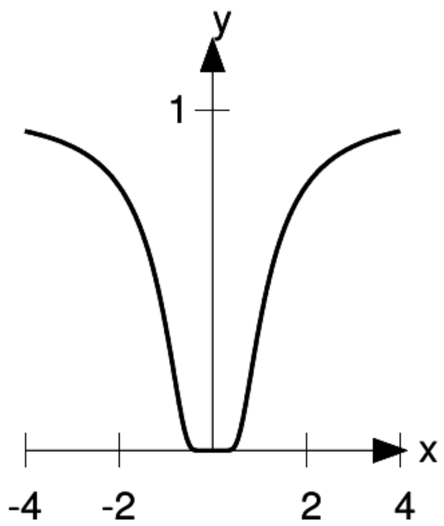


**Fig. 7.5:**

e / The function $e^{-1/x^2}$ never converges to its Taylor series.

never converges to its Taylor series, except at x = 0. This is because the Taylor series for this function, evaluated around x = 0 is exactly zero! At x = 0, we have y = 0, dy / dx = 0, $d^2y/dx^2$ = 0, and so on for every derivative. The zero function matches the function y (x) and all its derivatives to all orders, and yet is useless as an approximation to y (x). The radius of convergence of the Taylor series is infinite, but it doesn't give correct results except at x = 0. The reason for this is that y was built by composing two functions, w (x) = -1/$x^2$ and y (w) = $e^w$. The function w is badly behaved at x = 0 because it blows up there. In particular, it doesn't have a well-defined Taylor series at x = 0

## Example 85

Find the Taylor series of $y$= sin $x$, evaluated around $x$= 0.

The first few derivatives are

$$\frac{d}{dx}sinx = cosx$$

$$\frac{d^2}{dx^2}sinx = -sinx$$

$$\frac{d^3}{dx^3}sinx = -cosx$$

$$\frac{d^4}{dx^4}sinx = sinx$$

$$\frac{d^5}{dx^5}sinx = cosx$$

We can see that there will be a cycle of sin, cos, -sin, and -cos, repeating indefinitely. Evaluating these derivatives at $x$= 0, we have 0, 1, 0,-1, . . . . All the even-order terms of the series are zero, and all the odd- order terms are ±1/$n$!. The result is

$$sinx = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

The linear term is the familiar small- angle approximation sin $x$≈$x$.

The radius of convergence of this series turns out to be infinite. Intuitively the reason for this is that the factorials grow extremely rapidly, so that the successive terms in the series eventually start diminish quickly, even for large values of $x$.

## Example 86

Suppose that we want to evaluate a limit of the form

$$\lim_{x \to 0} \frac{u(x)}{v(x)},$$

where $u(0) = v(0) = 0$. L'Hoˆ pital's rule tells us that we can do this by taking derivatives on the top and bottom to form $u'/v'$, and that, if necessary, we can do more than one derivative, e.g.,$u''/v''$. This was proved using the mean value theorem. But if $u$ and $v$are both functions that converge to their Taylor series, then it is much easier to see why this works. For ex- ample, suppose that their Taylor series both have vanishing constant and linear terms, so that $u$= $a$ $x^2$ + ...and $v$= $b$$x^2$+ .... Then $u''$= 2$a$+ ..., and $v''$ = 2$b$+ ....

A function's Taylor series doesn't have to be evaluated around $x$=0. The Taylor series around some other center $x$= $c$is given by

$$T_c(x) = \sum_{n=0}^{\infty} a_n(x-c))^n$$

where

$$\frac{a_n}{n!} = \frac{d^n y}{dx^n}|_{x=c}$$

To see that this is the right generalization, we can do a change of variable, defining a new function *g*(*x*) = *f*(*x*−*c*). The radius of convergence is to be measured from the center *c*rather than from 0.

## Example 87

Find the Taylor series of ln *x*, evaluated around *x*= 1.

Evaluating a few derivatives, we get

$$\frac{d}{dx} Inx = x^{-1}$$
$$\frac{d^2}{dx^2} Inx = -x^{-2}$$
$$\frac{d^3}{dx^3} Inx = 2x^{-3}$$
$$\frac{d^4}{dx^4} Inx = -6x^{-4}$$

Note that evaluating these at *x*= 0 wouldn't have worked, since division by zero is undefined; this is because ln *x*blows up to negative infinity at *x* = 0. Evaluating them at *x*= 1, we find that the *nth*derivative equals ±(*n*−1)!, so the coefficients of the Taylor series are ±(*n*−1)!/*n*! = ±1/*n*, except for the *n*= 0 term, which is zero because ln 1 = 0. The resulting series is

$$Inx = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots$$

We can predict that its radius of convergence can't be any greater than 1, because ln *x*blows up at 0, which is at a distance of 1 from 1.

# 7.5 Problems

## 7.5.1 Problem 7.1



Modify the Weierstrass definition of the limit to apply to infinite sequences. Solutions for chapter 7 (Page 212)

## 7.5.2 Problem 7.2



(a) Prove that the infinite se ries 1 −1 + 1 −1 + 1 −1 + ...does not converge to any limit, using the generalization of the Weier- strass limit found in problem 1.
(b) Criticize the following argument. The series given in part a equals zero, because addition is associative, so we can rewrite it as (1 −1) + (1 −1) + (1 −1) + ...

Solutions for chapter 7 (Page 212)

## 7.5.3 Problem 7.3



Use the integral test to prove the convergence of the geometric series for $0 < x < 1$...

Solutions for chapter 7 (Page 212)

## 7.5.4 Problem 7.4



Determine the convergence or divergence of the following series.
(a) 1 + 1⁄22 + 1⁄32 + ...
(b) 1⁄ln ln 3−1⁄ln ln 6+1⁄ln ln 9−1= ln ln 12 +...
(c)

$$\frac{1}{In2}+\frac{1}{(In2)(In3)}+\frac{1}{(In2)(In3)(In4)}+\ldots$$

(d)

$$\frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

Solutions for chapter 7 (Page 212)

## 7.5.5 Problem 7.5



Give an example of a series for which the ratio test is inconclusive.

Solutions for chapter 7 (Page 212)

## 7.5.6 Problem 7.6



Find the Taylor series expansion of cos *x*around *x*= 0. Check your work by combining the first two terms of this series with the first term of the sine function from example 85 on page 112 to verify that the trig identity $sin^2 x + cos^2 x = 1$ holds for terms up to order $x^2$.

## 7.5.7 Problem 7.7



In classical physics, the kinetic energy *K*of an object of mass *m* moving at velocity *v*is given by *K*= $\frac{1}{2}mv^2$ . For example, if a car is to start from a stoplight and then accelerate up to *v*, this is the theoretical minimum amount of energy that would have to be used up by burning gasoline. (In reality, a car's engine is not 100% efficient, so the amount of gas burned is greater.) Einstein's theory of relativity states that the correct equation is actually

$$K = (\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}})mc^2$$

where *c* is the speed of light. The fact that it diverges as *v*→*c* is interpreted to mean that no object can be accelerated to the speed of light.2 2 X (4*k*)!(1103 + 26390*k*) Expand *K* in a Taylor series, and show that the first nonvanishing term is equal to the classical expression. This means that for velocities that are small compared to the speed of light, the classical expression is a good approximation, and Einstein's theory does not contradict any of the prior empirical evidence from which the classical expression was inferred.

## 7.5.8 Problem 7.8



Expand $(1 + x)^{1/3}$ in a Taylor series around *x* = 0. The value *x*= 28 lies outside this series' radius of convergence, but we can nevertheless use it to extract the cube root of 28 by recognizing that $28^{1/3} = 3(28/27)^{1/3}$. Calculate the root to four significant figures of precision, and check it in the obvious way.

## 7.5.9 Problem 7.9



Find the Taylor series expansion of $log_2 x$ around *x*= 1, and use it to evaluate $log_2$ 1.0595 to four significant figures of precision. Check your result by using the fact that 1.0595 is approximately the twelfth root of 2. This number is the ratio of the frequencies of two successive notes of the chromatic scale in music, e.g., C and D-flat.

## 7.5.10 Problem 7.10



In free fall, the acceleration will not be exactly constant, due to air resistance. For example, a skydiver does not speed up indefinitely until opening her chute, but rather approaches a certain maxi- mum velocity at which the upward force of air resistance cancels out the force of gravity. If an object is dropped from a height *h*, and the time it takes to reach the ground is used to measure the acceleration of gravity, *g*, then the relative error in the result due to air resistance is [1] (http://www.opentextbooks.org.hk/ditatopic/33508#)

$$E = \frac{g - g_{vacuum}}{g}$$
$$= 1 - \frac{2b}{In^2(e^b + \sqrt{e^{2b} - 1})}$$

where *b* = *h/A*, and *A*is a constant that depends on the size, shape, and mass of the object, and the density of the air. (For a sphere of mass *m*and diameter *d*dropping in air, *A*= 4.11$m/_{d}2$ . Cf. problem 20, p. 49.) Evaluate the constant and linear terms of the Taylor series for the function *E*(*b*).

## 7.5.11 Problem 7.11



(a) Prove that the convergence of an infinite series is un- affected by omitting some initial terms. (b) Similarly, prove that convergence is unaffected by multiplying all the terms by some constant factor.

## 7.5.12 Problem 7.12



The identity

$$\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}$$

is known as the “Sophomore’s dream,” because at first glance it looks like the kind of plausible but false statement that someone would naively dream up. Verify it numerically by machine computation.

### 7.5.13 Problem 7.13



Does sin *x*+ sin sin *x* +sin sin sin *x*+ ...converge? .

Solutions for chapter 7 (Page 212)

### 7.5.14 Problem 7.14



Evaluate

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots$$

Solutions for chapter 7 (Page 212)

### 7.5.15 Problem 7.15



Evaluate

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1+1/n!}$$

to six decimal places.

### 7.5.16 Problem 7.16



Euler was the first to prove

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$$

This problem had defeated other great mathematicians of his time, and was famous enough to be given a special name, the Basel problem. Here we present an argument based closely on Euler’s and pose the problem of how to exploit Euler’s technique further in order to prove

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$$

From the Taylor series for the sine function, we nd the related series

$$f(x)=\frac{sin\sqrt{x}}{\sqrt{x}}=1-\frac{x}{-3}+\frac{x^2}{5!}$$

The partial sums of this series are polynomials that approximate f for small values of x. If such a polynomial were exact rather than approximate, then it would have zeroes at x = $\pi^2, 4\pi^2, 9\pi^2, \ldots$, and we could write it as the product of its linear factors. Euler assumed, without any more rigorous proof, that this factorization procedure could be extended to the in nite series, so that f could be represented as the in nite product

$$f(x) - (1 - \frac{x}{\pi^2})(1 - \frac{x}{4\pi^2})\ldots$$

By multiplying this out and equating its linear term to that of the Taylor series, we nd the claimed result. Extend this procedure to the $x^2$ term and prove the result claimed for the sum of the inverse fourth powers of the integers. (The sums with odd exponents 3 are much harder, and relatively little is known about them. The sum of the inverse cubes is known as Apery's constant.)

## 7.5.17 Problem 7.17



Does

$$\int_0^\infty sin(x^2)dx$$

converge, or not?

Solutions for chapter 7 (Page 212)

## 7.5.18 Problem 7.18



Evaluates

$$\lim_{n\to\infty} cos(\pi\sqrt{n^2 - n})$$

where *n* is an integer.

## 7.5.19 Problem 7.19



Determine the convergence of the series

$$\sum_{n=0}^{\infty} n^2 2^{-n}$$

and if it converges, evaluate it.

Solutions for chapter 7 (Page 212)

## 7.5.20 Problem 7.20



Determine the convergence of the series

$$\sum_{n=0}^{\infty} n^2 2^{-n}$$

and if it converges, evaluate it.

Solutions for chapter 7 (Page 212)

## 7.5.21 Problem 7.21



For what integer values of $p$ should we expect the series

$$\sum_{n=1}^{\infty} \frac{| \, cos n \, |^n}{n^p}$$

to converge? A rigorous proof is very difficult and may even be an open problem, but it is relatively straightforward to give a convincing argument.

Solutions for chapter 7 (Page 212)

# Chapter 8 Complex number techniques

## 8.1 Review of complex numbers



For a more detailed treatment of complex numbers, see ch.3 of James Nearing's free book at http://www.physics.miami.edu/ nearing/mathmethods/. (http://www.physics.miami.edu/%20nearing/mathmethods/.)

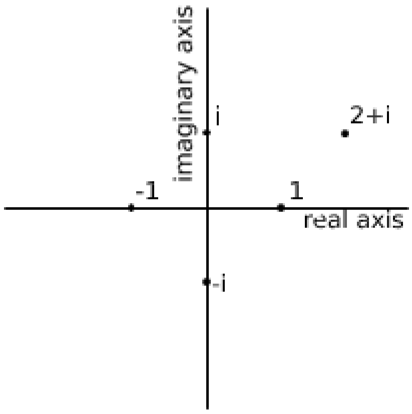


**Fig. 8.1: a / Visualizing complex numbers as points in a plane.**

We assume there is a number, *i*, such that $i^2 = -1$. The square roots of −1 are then *i* and −*i*. (In electrical engineering work, where *i*stands for current, *j* is sometimes used instead.) This gives rise to a number system, called the complex numbers, containing the real numbers as a subset.

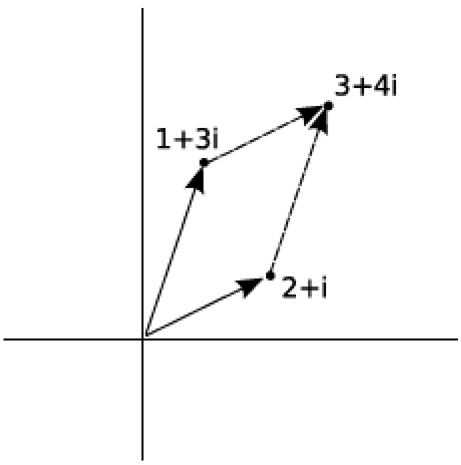


**Fig. 8.2: b / Addition of complex numbers is just like addition of vectors, although the real and imaginary axes don't actually represent directions in space.**

Any complex number *z*can be written in the form *z*= *a*+ *bi*, where *a*and *b*are real, and *a*and *b* ae then referred to as the real and imaginary parts of *z*. A number with a zero real part is called an imaginary number. The complex numbers can be visualized as a plane, figure a, with the real number line placed horizontally like the *x*axis of the familiar *x*−*y*plane, and the imaginary numbers running along the *y* axis. The complex numbers are complete in a way that the real numbers aren't: every nonzero complex number has two square roots. For example, 1 is a real number, so it is also a member of the complex numbers, and its square roots are −1 and 1. Like wise, −1 has square roots *i*and −*i*, and the number *i*has square roots $1/\sqrt{2}+i\sqrt{2}$ and $-1/\sqrt{2}-i\sqrt{2}$

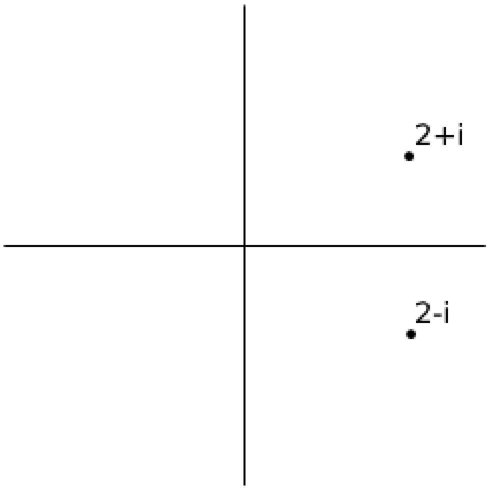


**Fig. 8.3: c / A complex number and its conjugate.**

Complex numbers can be added and subtracted by adding or subtracting their real and imaginary parts, figure b. Geometrically, this is the same as vector addition.

The complex numbers *a*+ *bi*and *a*−*bi*, lying at equal distances above and below the real axis, are called complex conjugates. The results of the quadratic formula are either both real, or complex conjugates of each other. The complex conjugate of a number *z*is notated as $\bar{z}$ or $z^*$.

The complex numbers obey all the same rules of arithmetic as the reals, except that they can't be ordered along a single line. That is, it's not possible to say whether one complex number is greater than another. We can compare them in terms of their magnitudes (their distances from the origin), but two distinct complex numbers may have the same magnitude, so, for example, we can't say whether 1 is greater than *i*or *i*is greater than 1.

## Example 88

Prove that $1/\sqrt{2}+i/\sqrt{2}$ is a square root of $i$.

Our proof can use any ordinary rules of arithmetic, except for ordering.

$$\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^2 = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}} = \frac{1}{2}(1+i+i-1) = i$$

Example 88 showed one method of multiplying complex numbers. However, there is another nice interpretation of complex multiplication. We define the argument of a complex number, figure d, as its angle in the complex plane, measured counterclockwise from the positive real axis. Multiplying two complex numbers then corresponds to multiplying their magnitudes, and adding their arguments, figure e.

# Self-Check

Using this interpretation of multiplication, how could you find the square roots of a complex number?

Answers to self-checks for chapter 8 (Page 216)

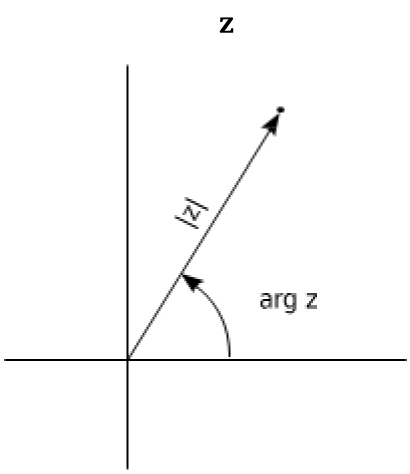


**Fig. 8.4: d / A complex number can be de- scribed in terms of its magnitude and argument.**

# Example 89

The magnitude $|z|$of a complex number $z$ obeys the identity $|z|^2 = z\bar{z}$. To prove this, we first note that $\bar{z}$ has the same magnitude as $z$, since flip- ping it to the other side of the real axis doesn't change its distance from the origin. Multiplying $z$by $\bar{z}$ gives a result whose magnitude is found by multiplying their magnitudes, so the magnitudez of $z\bar{z}$ must therefore equal $|z|^2$. Now we just have to prove that $z\bar{z}$ is a positive real number. But if, for example, $z$lies counterclockwise from the real axis, then $\bar{z}$ lies clockwise from it. If $z$ has a positive argument, then $\bar{z}$ has a negative one, or vice-versa. The sum of their arguments is there- fore zero, so the result has an argument of zero, and is on the positive real axis. [1]

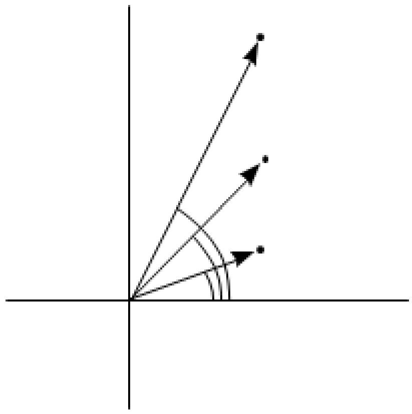

**Fig. 8.5: e / The argument of uv is the sum of the arguments of u and v .**

This whole system was built up in order to make every number have square roots. What about cube roots, fourth roots, and so on? Does it get even more weird when you want to do those as well? No. The complex number system we've already discussed is sufficient to handle all of them. The nicest way of thinking about it is in terms of roots of polynomials. In the real number system, the polynomial $x^2-1$ has two roots, i.e., two values of $x$(plus and minus one) that we can plug in to the polynomial and get zero. Because it has these two real roots, we can rewrite the polynomial as $(x-1)(x+1)$. However, the polynomial $x^2+1$ has no real roots. It's ugly that in the real number system, some second order polynomials have two roots, and can be factored, while others can't. In the complex number system, they all can. For instance, $x2 + 1$ has roots $i$ and $-i$, and can be factored as $(x-i)(x+i)$. In general, the fundamental theorem of algebra states that in the complex number system, any nth-order polynomial can be factored completely into $n$ linear factors, and we can also say that it has $n$ complex roots, with the understanding that some of the roots may be the same. For instance, the fourth- order polynomial $x^4+x^2$ can be factored as $(x-i)(x+i)(x-0)(x-0)$, and we say that it has four roots, $i$, $-i$, 0, and 0, two of which happen to be the same. This is a sensible way to think about it, because in real life, numbers are al- ways approximations anyway, and if we make tiny, random changes to the coefficients of this polynomial, it will have four distinct roots, of which two just happen to be very close to zero. I've given a proof of the fundamental theorem of algebra on page 162.

1. I cheated a little. If z's argument is 30 degrees, then we could say $\bar{z}$'s was −30, but we could also call it 330. That's OK, because 330+30 gives 360, and an argument of 360 is the same as an argument of zero.

## 8.2 Euler's formula



Having expanded our horizons to include the complex numbers, it's natural to want to extend functions we knew and loved from the world of real numbers so that they can also operate on complex numbers. The only really natural way to do this in general is to use Taylor series. A particularly beautiful thing happens with the functions $e^x$, sin *x*, and cos *x*:

$$e^x = 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$
$$cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$$
$$sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$

If *x*= *iφ* is an imaginary number, we have

$e^{i\varphi}$ = cos *φ*+ *i* sin *φ* ,

a result known as Euler's formula. The geometrical interpretation in the complex plane is shown in figure f.

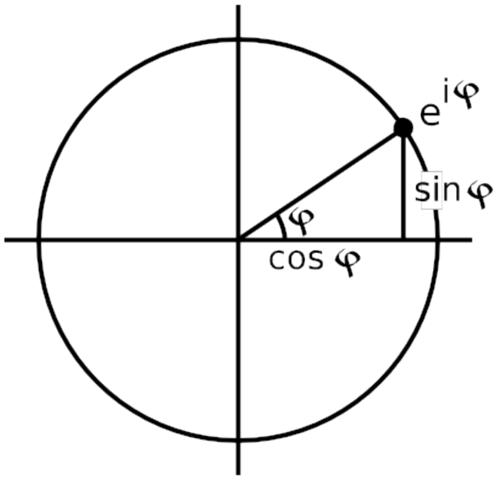


**Fig. 8.6:**

**Fig. 8.7: f / The complex number ei φ lies on the unit circle.**

Although the result may seem like something out of a freak show at first, applying the definition [2] of the exponential function makes it clear how natural it is:

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

When *x*= *iφ* is imaginary, the quantity (1 + *i φ* /*n*) represents a number lying just above 1 in the complex plane. For large *n*, (1 + *i φ* /*n*) becomes very close to the unit circle, and its argument is the small angle *φ* / *n*. Raising this number to the nth power multiplies its argument by *n*, giving a number with an argument of *φ*.

2. See page 151 for an explanation of where this definition comes from and why it makes sense.

**Fig. 8.8:**

**Fig. 8.9: g / Leonhard Euler (1707-1783)**

Euler's formula is used frequently in physics and engineering.

## Example 90

Write the sine and cosine functions in terms of exponentials.

Euler's formula for x= −iφ gives cosφ−i sinφ, since cos(−$\theta$) = cos$\theta$ , and sin(−$\theta$) = - sin$\theta$ .

$$cosX = \frac{e^{ix} + e^{-ix}}{2}$$
$$sinX = \frac{e^{ix} + e^{-ix}}{2i}$$

## Example 91

Evaluate

$$\int e^x cosx dx$$

Problem 5.15 (Page 120) suggested a special-purpose trick for doing this integral. An approach that doesn't rely on tricks is to rewrite the cosine in terms of exponentials:

$$\int e^x (\frac{e^{ix} + e^{-ix}}{2}) dx$$
$$\frac{1}{2}\int (e^{(1+i)x} + e^{(1-i)x}) dx$$
$$\frac{1}{2}(\frac{e^{(1+i)x}}{1+i} + \frac{e^{(1-i)x}}{1-i}) + c$$

Since this result is the integral of a real-valued function, we'd like it to be real, and in fact it is, since the first and second terms are complex conjugates of one another. If we wanted to, we could use Euler's theorem to convert it back to a manifestly real result. [3]

## Example 92

Euler found the equation

$$\pi = 20tan^{-1}\frac{1}{7} + 8tan^{-1}\frac{3}{79}$$

which allowed the computation of $\pi$ to high precision in the era before elec- tronic calculators, since the Taylor series for the inverse tangent converges rapidly for small inputs. A cute way of proving the validity of the equation is to calculate

$$(7+i)^{20}(79+3i)^{8}$$

as follows in Yacas:

```
(7+I)^20*(79+3*I)^8;
```

*-1490116119384765625*

*00000000000000*

The fact that it is purely real, and has a negative real part, demonstrates that the quantity on the right side of the original equation equals $\pi + 2\pi n$, where $n$is an integer. Numerical estimation shows that $n$= 0. Although the proof was straightforward, it provides zero insight into how Euler figured it out in the first place!

# 8.3 Partial fractions revisited



Suppose we want to evaluate the integral

$$\int \frac{dx}{x^2+1}$$

by the method of partial fractions. The quadratic formula tells us that the roots are $i$and $-i$, setting

$1/(x^2+1) = a/(x+i) + B/(x-i)$

gives A = i/2 and B = -i/2, so

3. In general, the use of complex number techniques to do an integral could result in a complex number, but that complex number would be a constant, which could be subsumed within the usual constant of integration.

$$\int \frac{dx}{x^2+1} = \frac{i}{2}\int \frac{dx}{x+i}$$
$$= \frac{i}{2}\int \frac{dx}{x-i}$$
$$= \frac{i}{2} In(x+i)$$
$$= \frac{i}{2} In(x-i)$$
$$= \frac{i}{2} In\frac{x+i}{x-i}$$

The attractive thing about this approach, compared with the method used on page 88, is that it doesn't require any tricks. If you came across this integral ten years from now, you could pull out your old calculus book, flip through it, and say, "Oh, here we go, there's a way to integrate one over a polynomial — partial fractions." On the other hand, it's odd that we started out trying to evaluate an integral that had nothing but real numbers, and came out with an answer that isn't even obviously a real number.

But what about that expression (*x*+*i*)/(*x*−*i*)? Let's give it a name, *w*. The numerator and denominator are complex conjugates of one another. Since they have the same magnitude, we must have |*w*|= 1, i.e., *w*is a complex number that lies on the unit circle, the kind of complex number that Euler's formula refers to. The numerator has an argument of $tan^{-1}(1/x) = \pi/2 - tan^{-1}x$ , and the denominator has the same argument but with the opposite sign. Division means subtracting arguments, so arg $w = \pi - 2tan^{-1}x$. That means that the result can be rewritten using Euler's formula as

$$\int \frac{dx}{x^2+1} = \frac{i}{2} Ine^{i(\pi-2tan^{-1}x)}$$
$$= \frac{i}{2}\cdot i(\pi - 2tan^{-1}x)$$
$$= tan^{-1}x + c$$

In other words, it's the same result we found before, but found with- out the need for trickery.

## Example 93

Evaluate $\int$ *dx*/sin *x*.

This can be tackled by rewriting the sine function in terms of complex exponentials, changing variables to *u*= $e^{ix}$, and then using partial fractions.

$$
\begin{aligned}
\int \frac{dx}{sinx} &= -2i \int \frac{dx}{e^{ix} - e^{-ix}} \\
&= -2i \int \frac{du/iu}{u - 1/u} \\
&= -2i \int \frac{du}{u^2 - 1} \\
&= In(u-1) - In(u+1) + c \\
&= In\frac{e^{ix} - 1}{e^{ix} + 1} + c \\
&= In(-itan(x/2)) + c \\
&= Initan(x/2)) + c'
\end{aligned}
$$

## 8.4 Problems

### 8.4.1 Problem 8.1



Find arg *i*, arg(−*i*), and arg 37, where arg *z*denotes the argument of the complex number *z*.

### 8.4.2 Problem 8.2



Visualize the following multiplications in the complex plane using the interpretation of multiplication in terms of multiplying magnitudes and adding arguments: (*i*)(*i*) = −1, (*i*)(−*i*) = 1, (−*i*)(−*i*) = −1.

### 8.4.3 Problem 8.3



If we visualize *z*as a point in the complex plane, how should we visualize −*z*?

### 8.4.4 Problem 8.4



Find four different complex numbers *z*such that $z^4$ = 1.

## 8.4.5 Problem 8.5



Compute the following:

$|1+i|, \arg(1+i)$

$$|\frac{1}{1+i}|, arg(\frac{1}{1+i})$$

$$\frac{1}{1+i}$$

## 8.4.6 Problem 8.6



Write the function tan $x$ in terms of complex exponentials.

## 8.4.7 Problem 8.7



Evaluate $\int sin^3 x dx$ .

## 8.4.8 Problem 8.8



Use Euler's theorem to derive the addition theorems that express sin($a$+ $b$) and cos($a$+ $b$) in terms of the sines and cosines of $a$ and $b$.

Answers to self-checks for chapter 8 (Page 216)

## 8.4.9 Problem 8.9



Evaluate

$$\int_0^{\pi/2} cos x cos 2x dx$$

Answers to self-checks for chapter 8 (Page 216)

## 8.4.10 Problem 8.10



Find every complex number *z* such that $z^3 = 1$.

Answers to self-checks for chapter 8 (Page 216)

## 8.4.11 Problem 8.11



Factor the expression $x^3 - y^3$ into factors of the lowest possible order, using complex coefficients. (Hint: use the result of problem 10.) Then do the same using real coefficients.

Answers to self-checks for chapter 8 (Page 216)

## 8.4.12 Problem 8.12



Evaluate

$$\int \frac{dx}{x^3 - x^2 + 4x - 4}$$

## 8.4.13 Problem 8.13



Evaluate

$$\int e^{-ax} cos bx dx$$

## 8.4.14 Problem 8.14



Consider the equation*f*′(*x*) = *f*(*f*(*x*)). This is known as a differential equation: an equation that relates a function to its own derivatives. What is unusual about this differential equation is that the right-hand side involves the function nested inside itself. Given, for example, the value of *f*(0), we expect the solution of this equation to exist and to be uniquely defined for all values of *x*. That doesn't mean, however, that we can write down such a solution as a closed-form expression. Show that two closed-form expressions do exist, of the form $f(x) = ax^b$, and find the two values of *b*.

Answers to self-checks for chapter 8 (Page 216)

## 8.4.15 Problem 8.15



(a) Discuss how the integral

$$\int \frac{dx}{x^{10000} - 1}$$

could be evaluated, in principle, in closed form.

(b) See what happens when you try to evaluate it using computer software.

(c) Express it as a finite sum.

Answers to self-checks for chapter 8 (Page 216)

# Chapter 9 Iterated integrals

## 9.1 Integrals inside integrals



In various applications, you need to do integrals stuck inside other integrals. These are known as iterated integrals, or double integrals, triple integrals, etc. Similar concepts crop up all the time even when you're not doing calculus, so let's start by imagining such an example. Suppose you want to count how many squares there are on a chess board, and you don't know how to multiply eight times eight. You could start from the upper left, count eight squares across, then continue with the second row, and so on, until you how counted every square, giving the result of 64. In slightly more formal mathematical language, we could write the following recipe: for each row, *r*, from 1 to 8, con- sider the columns, *c*, from 1 to 8, and add one to the count for each one of them. Using the sigma notation, this becomes

$$\sum_{r=1}^{8}\sum_{c=1}^{8} 1$$

If you're familiar with computer programming, then you can think of this as a sum that could be calculated using a loop nested in- side another loop. To evaluate the result (again, assuming we don't know how to multiply, so we have to use brute force), we can first evaluate the inside sum, which equals 8, giving

$$\sum_{r=1}^{8} 8$$

Notice how the "dummy" variable *c*has disappeared. Finally we do the outside sum, over *r*, and find the result of 64.

Now imagine doing the same thing with the pixels on a TV screen. The electron beam sweeps across the screen, painting the pixels in each row, one at a time. This is really no different than the example of the chess board, but because the pixels are so small, you normally think of the image on a TV screen as continuous rather than discrete. This is the idea of an integral in calculus. Suppose we want to find the area of a rectangle of width *a* and height *b*, and we don't know that we can just multiply to get the area *ab*. The brute force way to do this is to break up the rectangle into a grid of infinitesimally small squares, each having width d*x*and height d*y*, and therefore the infinitesimal area d*A*= d*x*d*y*. For convenience, we'll imagine that the rectangle's lower left corner is at the origin. Then the area is given by this integral:

$$\begin{aligned} area &= \int_{y=0}^{b}\int_{x=0}^{a} dA \\ &= \int_{y=0}^{b}\int_{x=0}^{a} dxdy \end{aligned}$$

Notice how the leftmost integral sign, over *y*, and the rightmost differential, d*y*, act like bookends, or the pieces of bread on a sandwich. Inside them, we have the integral sign that runs over *x*, and the differential d*x*that matches it on the right. Finally, on the inner-most layer, we'd normally have the thing we're integrating, but here's it's 1, so I've omitted it. Writing the lower limits of the integrals with *x* = and *y*= helps to keep it straight which integral goes with with differential. The result is

$$\begin{aligned} area &= \int_{y=0}^{b} \int_{x=0}^{a} dA \\ &= \int_{y=0}^{b} \int_{x=0}^{a} dxdy \\ &= \int_{y=0}^{b} (\int_{x=0}^{a} dx)dy \\ &= \int_{y=0}^{b} ady \\ &= ab \end{aligned}$$

## Area of a triangle Example 94

Find the area of a 45-45-90 right triangle having legs *a*.

Let the triangle's hypotenuse run from the origin to the point (*a*, *a*), and let its legs run from the origin to (0, *a*), and then to (*a*, *a*). In other words, the triangle sits on top of its hypotenuse. Then the integral can be set up the same way as the one before, but for a particular value of *y*, values of *x*only run from 0 (on the *y*axis) to *y*(on the hypotenuse). We then have

$$\begin{aligned} area &= \int_{y=0}^{a} \int_{x=0}^{y} dA \\ &= \int_{y=0}^{a} \int_{x=0}^{y} dxdy \\ &= \int_{y=0}^{a} (\int_{x=0}^{y} dx)dy \\ &= \int_{y=0}^{a} ydy \\ &= \frac{1}{2}a^2 \end{aligned}$$

Note that in this example, because the upper end of the *x* values depends on the value of *y*, it makes a difference which order we do the integrals in. The $x$ integral has to be on the inside, and we have to do it first.

## Volume of a cube Example 95

Find the volume of a cube with sides of length $a$.

This is a three-dimensional example, so we'll have integrals nested three deep, and the thing we're integrating is the volume d$V$= d$x$d$y$d$z$.

$$\begin{aligned} volume &= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} dV \\ &= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} dxdydz \\ &= \int_{z=0}^{a}\int_{y=0}^{a} adydz \\ &= a\int_{z=0}^{a}\int_{y=0}^{a} dydz \\ &= a\int_{z=0}^{a} adz \\ &= a^2\int_{z=0}^{a} dz \\ &= a^3 \end{aligned}$$

## Area of a circle Example 96

Find the area of a circle.

To make it easy, let's find the area of a semicircle and then double it. Let the circle's radius be $r$, and let it be centered on the origin and bounded below by the xaxis. Then the curved edge is given by the equation $R^2 = x^2 + y^2, ory = \sqrt{R^2 - x^2}dydx$ . Since the

$y$ integral's limit depends on $x$, the $x$ integral has to be on the outside. The area is

$$\begin{aligned} area &= \int_{x=-R}^{r}\int_{y=0}^{\sqrt{R^2-x^2}} dydx \\ &= \int_{x=-R}^{r} \sqrt{R^2 - x^2}dx \\ &= \int_{x=-R}^{r} \sqrt{1-(x/R)^2}dx \end{aligned}$$

The definite integral equals $\pi$, as you can find using a trig substitution or simply by looking it up in a table, and the result is, as expected,

$\pi R^2/2$ for the area of the semicircle. Doubling it, we find the expected result of $\pi R^2$ for a full circle.

## 9.2 Applications



Up until now, the integrand of the innermost integral has always been 1, so we really could have done all the double integrals as single integrals. The following example is one in which you really need to do iterated integrals.

**Fig. 9.1: a / The famous tightrope walker Charles Blondin uses a long pole for its large moment of inertia.**

### Moments of inertia Example 97

The moment of inertia is a measure of how difficult it is to start an object rotating (or stop it). For example, tightrope walkers carry long poles because they want something with a big moment of inertia. The moment of inertia is defined by $I = \int R^2 \, dm$, where d$m$ is the mass of an infinitesimally small portion of the object, and $R$ is the distance from the axis of rotation.

To start with, let's do an example that doesn't require iterated integrals. Let's calculate the moment of inertia of a thin rod of mass $M$ and length $L$ about a line perpendicular to the rod and passing through its center.

$$I = \int R^2 dm$$

$$= \int_{-L/2}^{L/2} X^2 \frac{M}{L} dx$$

$$[r = | x |, so R^2 = x^2]$$

$$= \frac{1}{12} M L^2$$

Now let's do one that requires iterated integrals: the moment of inertia of a cube of side $b$, for rotation about an axis that passes through its center and is parallel to four of its faces. Let the origin be at the center of the cube, and let *x* be the rotation axi

$$\begin{aligned} I &= \int R^2 dm \\ &= \rho \int R^2 dV \\ &= \rho \int_{b/2}^{b/2} \int_{b/2}^{b/2} \int_{b/2}^{b/2} (y^2 + z^2) dx dy dz \\ &= \rho \int_{b/2}^{b/2} \int_{b/2}^{b/2} (y^2 + z^2) dy dz \end{aligned}$$

The fact that the last step is a trivial integral results from the symmetry of the problem. The integrand of the remaining double integral breaks down into two terms, each of which depends on only one of the variables, so we break it into two integrals,

$$\begin{aligned} I = &\rho b \int_{b/2}^{b/2} \int_{b/2}^{b/2} y^2 dy dz \\ &+ \rho b \int_{b/2}^{b/2} \int_{b/2}^{b/2} z^2 dy dz \end{aligned}$$

which we know have identical results. We therefore only need to evaluate one of them and double the result:

$$\begin{aligned} I &= 2\rho b \int_{b/2}^{b/2} \int_{b/2}^{b/2} z^2 dy dz \\ &= 2\rho b^2 \int_{b/2}^{b/2} z^2 dz \\ &= \frac{1}{6} \rho b^5 \\ &= \frac{1}{6} M b^5 \end{aligned}$$

## 9.3 Polar coordinates



**Fig. 9.2: b/ Rene Descartes (1596-1650)**

Philosopher and mathematician Ren´e Descartes originated the idea of describing plane geometry using ($x$, $y$) coordinates measured from a pair of perpendicular coordinate axes. These rectangular coordinates are known as Cartesian co- ordinates, in his honor.

As a logical extension of Descartes' idea, one can find different ways of defining coordinates on the plane, such as the polar coordinates in figure c. In polar coordinates, the differential of area, figure d can be written as d$a$= $R$d$R$d$\varphi$. The idea is that since d$R$and d$\varphi$are infinitesimally small, the shaded area in the figure is very nearly a rectangle, measuring d$R$is one dimension and $R$d$\varphi$in the other. (The latter follows from the definition of radian measure.)

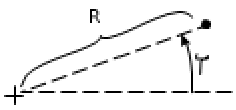


**Fig. 9.3: c / Polar coordinates.**

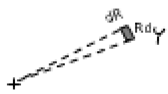

**Fig. 9.4: d / The differential of area in polar coordinates**

**Fig. 9.5:**

## Example 98

A disk has mass $M$and radius $b$. Find its moment of inertia for rotation about the axis passing perpendicularly through its center.

$$
\begin{aligned}
I &= \int R^2 dM \\
&= \int R^2 \frac{dM}{da} da \\
&= \int R^2 \frac{M}{\pi b^2} da \\
&= \frac{M}{\pi b^2} \int_{R=0}^{b} \int_{\o=0}^{2\pi} R^2 \cdot R d\o dR \\
&= \frac{M}{\pi b^2} \int_{R=0}^{b} R^3 \int_{\o=0}^{2\pi} \cdot R\o dR \\
&= \frac{2M}{b^2} \int_{R=0}^{b} R^3 dR \\
&= \frac{Mb^4}{2}
\end{aligned}
$$

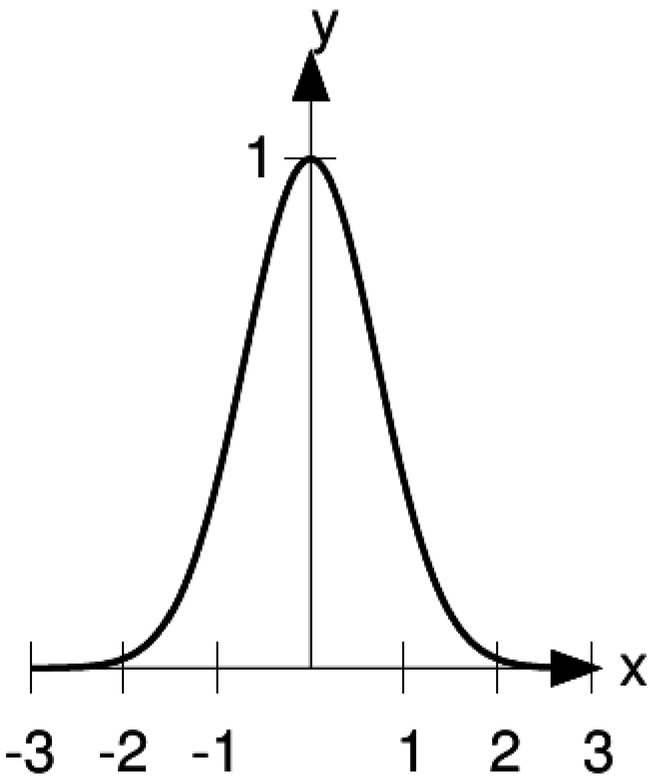


Fig. 9.6: e / The function e−x , example 99.

## Example 99

In statistics, the standard "bell curve" (also known as the normal distribution or Gaussian) is shaped like $e^{-x2}$. An area under this curve is proportional to the probability that *x*lies within a certain range. To fix the constant ofproportionality, we need to evaluate

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx,$$

which corresponds to a probability of

1. As discussed, the corresponding indefinite Integrals that can't be done (Page 115) in closed form. The definite integral from -∞ to +∞, however, can be evaluated by the following devious trick due to Poisson. We first write $I^2$ as a product of two copies of the integral.

$$I^2 = (\int_{-\infty}^{\infty} e^{-x^2} dx)(\int_{-\infty}^{\infty} e^{-x^2} dx)$$

Since the variable of integration xis a "dummy" variable, we can choose it to be any letter of the alphabet. Let's change the second one to *y*:

$$I^2 = (\int_{-\infty}^{\infty} e^{-x^2} dx)(\int_{-\infty}^{\infty} e^{-x^2} dy)$$

This is in principle a pointless and trivial change, but it suggests visualizing the right-hand side in the Cartesian plane, and considering it as the integral of a single function that depends on both *x* and *y*:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{-y^2} e^{-x^2}) dx dy$$

Switching to polar coordinates, we have

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} e^{-R^2} R dR d\o \\ &= 2\pi \int_0^{\infty} e^{-R^2} R dR, \end{aligned}$$

which can be done using the substitution *u*= $R^2$, d*u*= 2*R*d*R*:

$$\begin{aligned} I^2 &= 2\pi \int_0^{\infty} e^{-u} (du/2)) \\ &= \pi \\ I &= \sqrt{\pi} \end{aligned}$$

## 9.4 Spherical and cylindrical coordinates



In cylindrical coordinates ($R$, $\varphi$, $z$), $z$measures distance along the axis, $R$measures distance from the axis, and $\varphi$ is an angle that wraps around the axis.

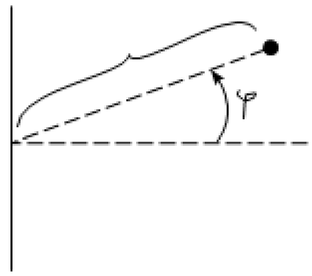


**Fig. 9.7: f / Cylindrical coordinates**

The differential of volume in cylindrical coordinates can be written as d$v$ = $R$d$R$d$z$d$\varphi$. This follows from adding a third dimension, along the $z$axis, to the rectangle in figure d.

### Example 100

Show that the expression for d$v$has the right units.

Angles are unitless, since the definition of radian measure involves a distance divided by a distance. Therefore the only factors in the expression that have units are $R$, d$R$, and d$z$. If these three factors are measured, say, in meters, then their product has units of cubic meters, which is correct for a volume.

### Example 101

Find the volume of a cone whose height is $h$and whose base has radius $b$.

Let's plan on putting the $z$integral on the outside of the sandwich. That means we need to express the radius *rmax* of the cone in terms of $z$. This comes out nice and simple if we imagine the cone upside down, with its tip at the origin. Then since we have $r_{max}$ ($z$= 0) = 0, and $r_{max}$ ($h$) = $b$, evidently $r_{max}$ = $zb/h$.

$$v = \int dv$$

As a check, we note that the answer has units of volume. This is the classical result, known by the ancient Egyptians, that a cone has one third the volume of its enclosing cylinder.

In spherical coordinates $(r, \theta, \varphi)$, the coordinate $r$measures the distance from the origin, and $\theta$and $\varphi$ are analogous to latitude and longitude, except that $\theta$is measured down from the pole rather than from the equator.

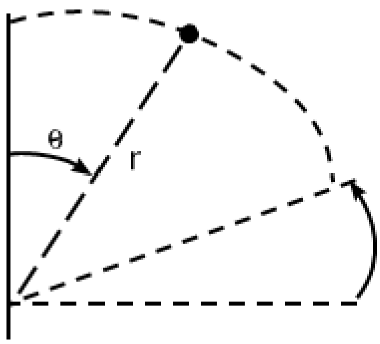


**Fig. 9.8: g/ Spherical coordinates**

The differential of volume in spherical coordinates is $dv = r^2 \sin \theta dr d\theta d\varphi$.

## Example 102

Find the volume of a sphere.

$$\begin{aligned} v &= \int dv \\ &= \int_{\theta=0}^{\pi} \int_{r=0}^{r=b} \int_{\phi=0}^{2\pi} r^2 sin\theta d\phi d\theta \\ &= 2\pi \int_{\theta=0}^{\pi} \int_{r=0}^{r=b} r^2 sin\theta\, dr d\theta \\ &= 2\pi \cdot \frac{b^3}{3} \int_{\theta=0}^{\pi} sin\theta d\theta \\ &= \frac{4\pi b^3}{3} \end{aligned}$$

## 9.5 Problems

### 9.5.1 Problem 9.1



Pascal's snail (named after E´ tienne Pascal, father of Blaise Pascal) is the shape shown in the figure, defined by $R= b(1 + \cos \theta)$ in polar coordinates.

(a) Make a rough visual estimate of its area from the figure.

(b) Find its area exactly, and check against your result from part a.

(c) Show that your answer has the right units. [Thompson, 1919]

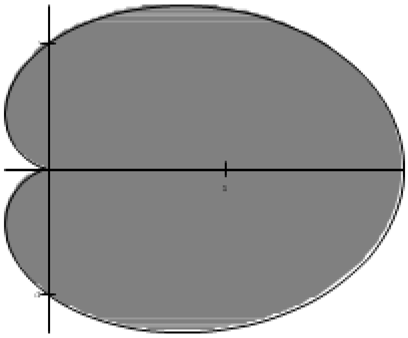

**Fig. 9.9: Problem 1: Pascal's snail with b = 1.**

## 9.5.2 Problem 9.2



A cone with a curved base is $r \le b[1 + c(\cos 2\theta - k)]$ ,defined by $r \le b$ and $\theta \le \pi/4$ in spherical coordinates.(a) Find its volume.(b) Show that your answer has the right units.

## 9.5.3 Problem 9.3



Find the moment of inertia of a sphere for rotation about an axis passing through its center.

## 9.5.4 Problem 9.4



A jump-rope swinging in circles has the shape of a sine function. Find the volume enclosed by the swinging rope, in terms of the radius *b*of the circle at the rope's fattest point, and the straight-line distance `between the ends.

## 9.5.5 Problem 9.5



A curvy-sided cone is defined in cylindrical coordinates by $0 \le z \le h$ and $R \le kz^2$. (a) What units are implied for the constant $k$? (b)Find the volume of the shape. (c) Check that your answer to b has the right units.

### 9.5.6 Problem 9.6



The discovery of nuclear fission was originally explained by modeling the atomic nucleus as a drop of liquid. Like a water balloon, the drop could spin or vibrate, and if the motion became sufficiently violent, the drop could split in half — undergo fission. It was later learned that even the nuclei in matter under ordinary conditions are often not spherical but deformed, typically with an elongated ellipsoidal shape like an American football. One simple way of describing such a shape is with the equation

$$r \leq b[1 + c(cos^2\theta - k)],$$

where *c*= 0 for a sphere, *c*>0 for an elongated shape, and *c*<0 for a flattened one. Usually for nuclei in ordinary matter, *c*ranges from about 0 to +0.2. The constant *k* is introduced because without it, a change in *c*would entail not just a change in the shape of the nucleus, but a change in its volume as well. Observations show, on the contrary, that the nuclear fluid is highly incompressible, just like ordinary water, so the volume of the nucleus is not expected to change significantly, even in violent processes like fission. Calculate the volume of the nucleus, throwing away terms of order $c^2$ or higher, and show that *k*= 1/3 is required in order to keep the volume constant.

### 9.5.7 Problem 9.7



This problem is a continuation of problem 6, and assumes the result of that problem is already known. The $nucleus^{168}$ Er has the type of elongated ellipsoidal shape described in that problem, with *c* >0. Its mass is $2.8X10^{-25}$ kg, it is observed to have a moment of inertia of$2.62X10^{-54}$ kg · $m^2$ for end-over-end rotation, and its shape is believed to be described by $b \approx 6X10^{-15}$ m and *c* ≈ 0.2. Assuming that it rotated rigidly, the usual equation for the moment of inertia could be applicable, but it may rotate more like a water balloon, in which case its moment of inertia would be significantly less because not all the mass would actually flow. Test which type of rotation it is by calculating its moment of inertia for end-over-end rotation and comparing with the observed moment of inertia.

### 9.5.8 Problem 9.8



Von K´arm´an found empirically that when a fluid flows turbulently through a cylindrical pipe, the velocity of flow *v*varies according to the “1/7 power law,” $v/v_o = (1 - r/R)^{1/7}$ , where $v_o$ is the velocity at the center of the pipe, *R*is the radius of the pipe, and *r*is the distance from the axis. Find the average velocity at which water is transported through the pipe.

# Detours

## Formal definition of the tangent line



Given a function $x(t)$, consider any point P = $(a, x(a))$ on its graph. Let the function $\ell(t)$ be a line passing through P. We say that $\ell$ cuts through $x$ at P if there exists some real number $d>0$ such that the graph of $\ell$ is on one side of the graph of $x$ for all $a-d<t<a$, and is on the other side for all $a<t<a+d$.

Definition (Marsden [1]): A line $\ell$ through P is said to be the line tangent to $x$ at P if all lines through P with slopes less than that of $\ell$ cut through $x$ in one direction, while all lines with slopes greater than P's cut through it in the opposite direction.

The reason for the complication in the definition is that there are cases in which the function is smooth and well-behaved throughout a certain region, but for a certain point P in that region, all lines through P cut through P. For example, the function $x(t) = t^3$ is blessed everywhere with lines that don't cut through it — everywhere, that is, except at $t= 0$, which is an inflection point (p. 17). Our definition fills in the "gap tooth" in the derivative function in the obvious way.

### Example 103

As an example, we demonstrate that the derivative of $t^3$ is zero where it passes through the origin. Define the line $\ell(t) = bt$ with slope $b$, passing through the origin. For $b<0$, $\ell$ cuts the graph of $t^3$ once at the origin, going down and to the right. For $b>0$, $\ell$ cuts the graph of $t^3$ in three places, at $t= 0$ and $\pm\sqrt{b}$.

Picking any positive value of $d$ less than $\sqrt{b}$, we find that $\ell$ cuts the graph at the origin, going up and to the right. Therefore $b= 0$ gives the tangent line at the origin.

## Derivatives of polynomials



Some ideas in this proof are due to Tom Goodwillie. Theorem: For $n= 0, 1, 2, \ldots$, the derivative of the function $x$ defined by $x(t) = t^n$ is $\dot{x} = nt^{n-1}$.

The results for $n= 0$ and 1 hold by direct application of the definition of the derivative.

1. Calculus Unlimited, by Jerrold Marsden and Alan Weinstein, http://resolver.caltech.edu/CaltechBOOK:1981.001

For $n>1$, it suffices to prove $\dot{x}(0) = 0$ and $\dot{x}(1) = n$, since the result for other nonzero values of $t$ then follows by the kind of scaling argument used on page 13 for the $n= 2$ case.

We use the following properties of the derivative, all of which follow immediately from its definition as the slope of the tangent line:

*Shift*. Shifting a function $x(t)$ horizontally to form a new function $x(t+c)$ gives a derivative at any newly shifted point that is the same as the derivative at the corresponding point on the unshifted graph.

*Flip*. Flipping the function $x(t)$ to form a new function $x(-t)$ negates its derivative at $t= 0$.

*Add*. The derivative of the sum or difference of two functions is the sum or difference of their derivatives.

For even $n$, $\dot{x}(0) = 0$ follows from the flip property, since $x(-t)$ is the same function as $x(t)$. For $n= 3, 5, \ldots$, we apply the definition of the derivative in the same manner as was done in the preceding section for

$n= 3$.

We now need to show that $\dot{x}(1) = n$. Define the function $u$ as

$$u(t) = x(t+ 1) - x(t)$$

$$= 1 + nt + \ldots \quad ,$$

where the second line follows from the binomial theorem, and. . . represents terms involving $t^2$ and higher powers. Since we've already established the results for $n= 0$ and 1, differentiation gives

$$\dot{u}(t) = n + \ldots \quad .$$

Now let's evaluate this at $t= 0$, where, as shown earlier, the terms represented by . . . all vanish. Applying the add and shift properties, we have

$$\dot{x}(1) - \dot{x}(0) = n \quad .$$

But since $\dot{x}(0) = 0$, this completes the proof.

Although this proof was for integer exponents $n\geq1$, the result is also true for any real value of $n$; see example 24 on p. 41.

## Details of the proof of the derivative of the sine function



Some ideas in this proof are due to Jerome Keisler (see references, p.201).

On page 28, I computed the derivative of sin $t$ to be cos $t$ as follows:

$$dx = \sin(t + dt) - \sin t \quad ,$$

$$= \sin t \cos dt$$

$+ \cos t \sin dt - \sin t$

$= \cos t dt + \ldots$ .

We want to prove prove that the error ". . . " introduced by the small- angle approximations really is of order $dt^2$ .

A quick and dirty way to check whether this is likely to be true is to use Inf to calculate sin(*t*+ d*t*) at some specific value of *t*. For example, at *t*= 1 we have this result:

: sin(1+d)

*(0.84147)+(0.54030)d*

*+(-0.42074)d^2+(-0.09006)d^3*

*+(0.03506)d^4*

The small-angle approximations give sin(1 + *d*) ≈sin 1 + (cos 1)*d*. The coefficients of the first two terms of the exact result are, as expected sin(1) = 0.84147 and cos(1) = 0.5403 ..., so although the small-angle approximations have introduced some errors, they involve only higher powers of d*t*, as claimed.

The demonstration with Inf has two shortcomings. One is that it only works for *t*= 1, but we need to prove that the result for all values of *t*. That doesn't mean that the check for *t*= 1 was useless. Even though a general mathematical statement about all numbers can never be *proved* by demonstrating specific examples for which it succeeds, a single counterexample suffices to *disprove*it. The check for *t*= 1 was worth doing, because if the first term had come out to be 0.88888, it would have immediately disproved our claim, thereby saving us from wasting hours attempting to prove something that wasn't true.

The other problem is that I've never explained how Inf calculates this kind of thing. The answer is that it uses something called a Taylor series, discussed in section 7.4. Using Inf here without knowing yet how Taylor series work is like using your calculator as a "black box" to extract the square root of √2 without knowing how it does it. Not knowing the inner workings of the black box makes the demonstration less than satisfying.

In any case, this preliminary check makes it sound like it's reasonable to go on and try to produce a real proof. We have

$$\sin(t + dt) = \sin t + \cos t dt - E ,$$

where the error *E*introduced by the approximations is

$$E = \sin t(1 - \cos dt)$$
$$+ \cos t(dt - \sin dt) .$$

Let the radius of the circle in figure a be one, so AD is cos d*t*and CD is sin d*t*. The area of the shaded pie slice is d*t/*2, and the area of triangle ABC is sin d*t/*2, so the error made in the approximation sin d*t*≈d*t*equals twice the area of the dish shape formed by line BC and arc BC. Therefore d*t*−sin d*t*is less than the area of rectangle CEBD. But CEBD has both an infinitesimal width and an infinitesimal height, so this error is of no more than order d*t*2 .

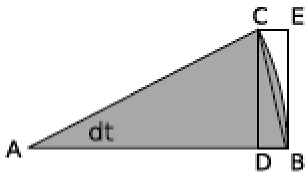


**Fig. 9.10: a / Geometrical interpretation of the error term.**

For the approximation cos d*t*≈1, the error (represented by BD) is $1 - cosdt = 1 - \sqrt{1 - sin^2 dt}$, which is less than $1 - \sqrt{1 - dt^2}$ , since sin d*t*<d*t*. Therefore this error is of order $dt^2$ .

# Formal statement of the transfer principle



I gave Safe use of infinitesimals (Page 32). The idea being expressed was that the phrases “for any” and “there exists” can only be used in phrases like “for any real number *x*” and “there exists a real number *y*such that. . . ” The transfer principle does not apply to statements like “there exists an integer *x* such that. . . ” or even “there exists a subset of the real numbers such that. . . ”

The way to state the transfer principle more rigorously is to get rid of the ambiguities of the English language by restricting ourselves to a well- defined language of mathematical symbols. This language has symbols *?* and *?*, meaning” for all” and” there exists,” and these are called quantifiers. A quantifier is always immediately followed by a variable, and then by a statement involving that variable. For example, suppose we want to say that a number greater than 1 exists. We can write the statement *?xx*>1, read as “there exists a number *x*such that *x*is greater than 1.” We don’t actually need to say “there exists a number *x in the set of real numbers* such that . . . ,” because our intention here is to make statements that can be translated back and forth between the reals and the hyper reals. In fact, we forbid this type of explicit reference to the domain to which the quantifiers apply. This restriction is described technically by saying that we’re only allowing *first-order logic*.

Quantifiers can be nested. For example, I can state the commutativity of addition as *?x?yx*+ *y*= *y*+ *x*, and the existence of additive inverses as *?x?yx*+ *y*= 0.

After the quantifier and the variable, we have some mathematical assertion, in which we’re allowed to use the symbols =, >, ×and + for the basic operations of arithmetic, and also parentheses and the logical operators ¬, *?*and *?*for “not,” “and,” and “or.” Although we will often find it convenient to use other symbols, such as 0, 1, −, /, ≤, =, etc., these are not strictly necessary. We use them only as a way of making the formulas more readable, with the understanding that they could be translated into the more basic symbols. For instance, I can restate *?xx*>1 as *?x?y?zyz*= *z?x*>*y*. The number *y*ends up just being a name for 1, because it’s the only number that will always satisfy *yz*= *z*.

Finally, these statements need to satisfy certain syntactic rules. For example, we can't have a string of symbols like *x*+ ×*y*, because the operators + and ×are supposed to have numbers on both sides.

A finite string of symbols satisfying all the above rules is called a well- formed formula (wff ) in first-order logic.

The transfer principle states that a wff is true on the real numbers if and only if it is true on the hyperreal numbers.

If you look in an elementary algebra textbook at the statement of all the elementary axioms of the real number system, such as commutativity of multiplication, associativity of addition, and so on, you'll see that they can all be expressed in terms of first-order logic, and therefore you can use them when manipulating hyperreal numbers. However, it's not possible to fully characterize the real number system without giving at least some further axioms that cannot be expressed in first order. There is more than one way to set up these additional axioms, but for example one common axiom to use is the Archimedean principle, which states that there is no number that is greater than 1, greater than 1 + 1, greater than 1 + 1 + 1, and so on. If we try to express this as a well-formed formula in first order logic, one attempt would be ¬*?xx*>1 *? x* >1 + 1 *? x* >1 + 1 + 1 ..., where the ...indicates that the string of symbols would have to go on forever. This doesn't work because a well-formed formula has to be a *finite* string of symbols. Another attempt would be *?x?n?*N *x* >*n*, where N means the set of integers. This one also fails to be a wff in first-order logic, because in first-order logic we're not allowed to explicitly refer to the domain of a quantifier. We conclude that the transfer principle does not necessarily apply to the Archimedean principle, and in fact the Archimedean principle is not true on the hyperreals, because they include numbers that are infinite.

Now that we have a thorough and rigorous understanding of what the transfer principle says, the next obvious question is why we should believe that it's true. This is discussed in the following section.

## Is the transfer principle true?



The preceding section stated the transfer principle in rigorous language. But why should we believe that it's true?

One approach would be to begin deducing things about the hyperreals, and see if we can deduce a contradiction. As a starting point, we can use the axioms of elementary algebra, because the transfer principle tells us that those apply to the hyperreals as well. Since we also assume that the Archimedean principle does *not*hold for the hyperreals, we can also base our reasoning on that, and therefore many of the things we can prove will be things that are true for the hyperreals, but false for the reals. This is essentially what mathematicians started doing immediately after Newton and Leibniz invented the calculus, and they were immediately successful in producing contradictions. However, they weren't using formally defined logical systems, and they hadn't stated anything as specific and rigorous as the transfer principle. In particular, they didn't understand the need for anything like our restriction of the transfer principle to first-order logic. If we could reach a contradiction based on the more modern, rigorous statement of the transfer principle, that would be a different matter. It would tell us that one of two things was true: either (1) the hyperreal number system lacks logical self- consistency, or (2) both the hyperreals and the reals lack self-consistency.

Abraham Robinson proved, however, around 1960 that the reals and the hyperreals have the same level of consistency: one is self-consistent if and only if the other is. In other words, if the hyperreals harbor a ticking logical time bomb, so do the reals. Since most mathematicians don't lose much sleep worrying about a lack of self-consistency in the real number system, this is generally taken as meaning that infinitesimals have been rehabilitated. In fact, it gives them an even higher level of respectability than they had in the era of Gauss and Euler, when they were widely used, but mathematicians knew a valid style of proof involving infinitesimals only because they'd slowly developed the right "Spidey sense."

But how in the world could Robinson have proved such a thing? It seems like a daunting task. There is an infinite number of possible logical trains of argument in mathematics. How could he have demonstrated, with a stroke of a pen, that *none*of them could ever lead to a contradiction (unless it indicated a contradiction lurking in the real number system as well)? Obviously it's not possible to check them all explicitly.

The way modern logicians prove such things is usually by using *models*. For an easy example of a model, consider Euclidean geometry. Euclid believed that the following four postulates [2] were all self-evident:

1. Let the following be postulated: to draw a straight line from any point to any point.
2. To extend a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles are equal to one another.

These postulates, which today we would call "axioms," played the same role with respect to Euclidean geometry that the elementary axioms of arithmetic play for the real number system.

Euclid also found that he needed a fifth postulate in order to prove many of his most important theorems, such as the Pythagorean theorem. I'll state a different axiom that turns out to be equivalent to it:

5\. *Playfair's version of the parallel postulate:* Given any infinite line L, and any point P not on that line, there exists a unique infinite line through P that never crosses L.

The ancients believed this to be less obviously self-evident than the first four, partly because if you were given the two lines, it could theoretically take an infinite amount

2. 2 modified slightly by me from a translation by T.L. Heath, 1925

of time to inspect them and verify that they never crossed, even at some very distant point. Euclid avoided even mentioning infinite lines in postulates 1-4, and he considered postulate 5 to be so much less intuitively appealing in comparison that he organized the *Elements* so that the first 28 propositions were those that could be proved without resorting to it. Continuing the analogy with the reals and hyperreals, the parallel postulate plays the role of the Archimedean principle: a statement about infinity that we don't feel quite so sure about.

For centuries, geometers tried to prove the parallel postulate from the first five. The trouble with this kind of thing was that it could be difficult to tell what was a valid proof and what wasn't. The postulates were written in an ambiguous human language, not a formal logical system. As an example of the kind of confusion that could result, suppose we assume the following postulate, 5′, in place of 5:

5′: Given any infinite line L, and any point P not on that line, every infinite line through P crosses L.

Postulate 5′plays the role for noneuclidean geometry that the negation of the Archimedean principle plays for the hyperreals. It tells us we're not in Kansas anymore. If a geometer can start from postulates 1-4 and 5′and arrive at a contradiction, then he's made significant progress toward proving that postulate 5 has to be true based on postulates 1-4. (He would also have to disprove another version of the postulate, in which there is more than one parallel through P.) For centuries, there have been reasonable-sounding arguments that seemed to give such a contradiction. For instance, it was proved that a geometry with 5′in it was one in which distances were limited to some finite maximum. This would appear to contradict postulate 3, since there would be a limit on the radius of a circle. But there's plenty of room for disagreement here, because the ancient Greeks didn't have any notion of a set of real numbers. For them, the thing we would call a number was simply a finite straight line (line segment) with a certain length. If postulate 3 says that we can make a circle given any radius, it's reasonable to interpret that as a statement that *given any finite straightline*as the specification of the radius, we can make the circle. There is then no contradiction, because the too-long radius can't be specified in the first place. This muddle is similar to the kind of confusion that reigned for centuries after Newton: did infinitesimals lead to contradictions?

In the 19th century, Lobachevsky and Bolyai came up with a version of Euclid's axioms that was more rigorously defined, and that was care- fully engineered to avoid the kinds of contradictions that had previously been discovered in noneuclidean geometry. This is analogous to the in- vention of the transfer principle and the realization that the restriction to first-order logic was necessary. Lobachevsky and Bolyai slaved away for year after year proving new results in noneuclidean geometry, won- dering whether they would ever reach a contradiction. Eventually they started to doubt that there were ever going to be contradictions, and finally they proved that the contradictions didn't exist.

The technique for proving consistency was to make a *model*of the noneuclidean system. Consider geometry done on the surface of a sphere. The word "line" in the axioms now has to be understood as referring to a great circle, i.e., one with the same radius as the sphere. The parallel postulate fails, because parallels don't exist: every great circle intersects every other great circle. One modification has to be made to the model in order to make it consistent with the first postulate. The constructions described in Euclid's postulates are tacitly assumed to be unique (and in more rigorous formulations are explicitly stated to be so). We want there to be a unique line defined by any two distinct points. This works fine on the sphere as long as the points aren't too far apart, but it fails if the points are antipodes, i.e., they lie at opposite sides of the sphere. For example, every line of longitude on the Earth's surface passes through both poles. The solution to this problem is to modify what we mean by "point." Points at each other's antipodes are considered to be the *same point*. (Or, equivalently, we can do geometry on a hemisphere, but agree that when we go off one edge, we "wrap around" to the opposite side.)

This spherical model obeys all the postulates of this particular system of noneuclidean geometry. But consider now that we constructed it *inside* a surrounding three-dimensional space in which the parallel postulate does hold. Now suppose we keep on proving theorems in this system of noneuclidean geometry, filling up page after page with proofs using words like "line," which we mentally associate with great circles on a certain sphere — and eventually we reach a contradiction. But now we can go back through our proofs, and in every place where the word "line" occurs we can cross it out with a red pencil and put in "great circle on this particular sphere." It would now be a proof about *Euclidean*geometry, and the contradiction would prove that *Euclidean*geometry lacked self-consistency. We therefore arrive at the result that if noneuclidean geometry is inconsistent, so is Euclidean geometry. Since nobody believes that Euclidean geometry is inconsistent, this is considered the moral equivalent of proving noneuclidean geometry to be consistent.

If you've been keeping the system of analogies in mind as you read this story, it should be clear what's coming next. If we want to prove that the hyperreals have the same consistency as the reals, we just have to construct a *model*of the hyperreals using the reals. This is done in detail elsewhere (see Stroyan and Mathforum.org in the references, p. 201). I'll just sketch the general idea. A hyperreal number is represented by an infinite sequence of real numbers. For example, the sequence

$$7, 7, 7, 7, \ldots$$

would be the hyperreal version of the number 7. A sequence like

$$1, 2, 3, \ldots$$

represents an infinite number, while

$$1, \frac{1}{2}, \frac{1}{3}, \ldots$$

is infinitesimal. All the arithmetic operations are defined by applying them to the corresponding members of the sequences. For example, the sum of the 7, 7, 7, . . . sequence and the 1, 2, 3, . . . sequence would be 8, 9, 10, . . . , which we interpret as a somewhat larger infinite number.

The big problem in this approach is how to compare hyperreals, because a comparison like <is supposed to give an answer that is either true or false. It's not supposed to give a hyperreal number as the result.

It's clear that 8, 9, 10, . . . is greater than 1, 1, 1, . . . , because every member of the first sequence is greater than every member of the sec- ond one. But is 8, 9, 10, . . . greater than 9, 9, 9, . . . ? We want the answer to be "yes," because we're thinking of the first one as an infinite number and the second one as the ordinary finite number 9. The first sequence is indeed greater than the second at almost every one of the infinite number of places at which they could be compared. The only place where it loses the contest is at the very first position, and the only spot where we get a tie is the second one. Essentially the idea is that we want to define a concept of what happens "almost everywhere" on some infinite list. If one thing happens in an infinite number of places and something else only happens at some finite number of spots, then the definition of "almost everywhere" is clear. What's harder is a comparison of something like these two sequences:

2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, ...

and

1, 3, 1, 1, 3, 1, 1, 1, 3, 1, 1, 1, 1, 3, ...

where the second sequence has longer and longer runs of ones interspersed between the threes. The two sequences are never equal at any position, so clearly they can't be considered to be equal as hyperreal numbers. But there is an infinite number of spots in which the first sequence is greater than the second, and likewise an infinite number in which it's less. It seems as though there are more in which it's greater, so we probably want to define the second sequence as being a hyperreal number that's less than 2. The problem is that it can be very difficult to write down an acceptable definition of this "almost everywhere" notion. The answer is very technical, and I won't go into it here, but it can be done. Because two sequences could be equal almost everywhere, we end up having to define a hyperreal number not as a particular sequence but as a *set* of sequences that are equal to each other almost everywhere.

With the construction of this model, it is possible to prove that the hyperreals have the same level of consistency as the reals.

## The transfer principle applied to functions



I told you not to worry about Safe use of infinitesimals (Page 32). But since you're reading this, you're obviously in need of more reassurance.

For some of these functions, the transfer principle straightforwardly guarantees that they work for hyperreals, have all the familiar proper- ties, and can be computed in the same way. For example, the following statement is in a suitable form to have the transfer principle applied to it: *For any real number x,x·x≥0.*Changing "real" to "hyperreal," we find out that the square of a hyperreal number is greater than or equal to zero, just like the square of a real number. Writing it as $x^2$ or calling it a square is just a matter of notation and terminology. The same applies to this statement: *For any real number x≥0, there exists a real number y such that* $y^2$ *= x.*

Applying the transfer function to it tells us that square roots can be defined for the hyperreals as well.

There's a problem, however, when we get to functions like sin *x* and $e^x$. If you look up the definition of the sine function in a trigonometry textbook, it will be defined geometrically, as the ratio of the lengths of two sides of a certain triangle. The transfer principle doesn't apply to geometry, only to arithmetic. It's not even obvious intuitively that it makes sense to define a sine function on the hyperreals. In an application like the differentiation of the sine function on page 28, we only had to take sines of hyperreal numbers that were infinitesimally close to real numbers, but if the sine is going to be a full-fledged function defined on the hyperreals, then we should be allowed, for example, to take the sine of an infinite number. What would that mean? If you take the sine of a number like a million or a billion on your calculator, you just get some apparently random result between −1 and 1. The sine function wiggles back and forth indefinitely as *x*gets bigger and bigger, never settling down to any specific limiting value. Apparently we could have sin *H*= 1 for a particular infinite *H*, and then sin(*H* + *π*/2) = 0, sin(*H* + *π*) = −1,

. . .

It turns out that the moral equivalent of the transfer function can indeed be applied to any function on the reals, yielding a function that is in some sense its natural "big brother" on the the hyperreals, but the consequences can be either disturbing or exhilirating depending on your tastes. For example, consider the function [*x*] that takes a real number *x*and rounds it down to the greatest integer that is less than or equal to to *x*, e.g., [3] = 3, and [*π*] = 3. This function, like any other real function, can be extended to the hyperreals, and that means that we can define the *hyperintegers*, the set of hyperreals that satisfy [*x*] = *x*. The hyperintegers include the integers as a subset, but they also include infinite numbers. This is likely to seem magical, or even unreasonable, if we come at the hyperreals from a purely axiomatic point of view. The extension of functions to the hyperreals seems much more natural in view of the construction of the hyperreals in terms of sequences given in the preceding section. For example, the sequence 1.3, 2.3, 3.3, 4.3, 5.3, ...represents an infinite number. If we apply the [*x*] function to it, we get 1, 2, 3, 4, 5, ..., which is an infinite integer.

# Proof of the chain rule



In the statement of the chain rule on page 37, I followed my usual custom of writing derivatives as d*y*/d*x*, when actually the derivative is the standard part, st(d*y*/d*x*). In more rigorous notation, the chain rule should be stated like this:

$$st(\frac{dz}{dx}) = st(\frac{dz}{dy})st(\frac{dy}{dx})$$

The transfer principle allows us to rewrite the left-hand side as st[(d*z*/d*y*)(d*y*/d*x*)], and then we can get the desired result using the identity st(*ab*) = st(*a*)st(*b*).

# Derivative of ex



All of the reasoning on page 39 would have applied equally well to any other exponential function with a different base, such as $2^x$ or $10^x$. Those functions would have different values of *c*, so if we want to deter- mine the value of *c*for the base-*e*case, we need to bring in the definition of *e*, or of the exponential function $e^x$, somehow.

We can take the definition of $e^x$ to be

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

The idea behind this relation is similar to the idea of compound interest. If the interest rate is 10%, compounded annually, then *x*= 0.1, and the balance grows by a factor (1 + *x*) = 1.1 in one year. If, instead, we want to compound the interest monthly, we can set the monthly interest rate to 0.1/12, and then the growth of the balance over a year is $(1 + x/12)^{12}$ = 1.1047, which is slightly larger because the interest from the earlier months itself accrues interest in the later months. Continuing this limiting process, we find $e^{1.1}$= 1.1052.

If *n* is large, then we have a good approximation to the base-*e* exponential, so let's differentiate this finite-*n*approximation and try to find an approximation to the derivative of $e^x$ . The chain rule tells is that the derivative of $(1 + x/n)^n$ is the derivative of the raising-to- the-nth-power function, multiplied by the derivative of the inside stuff,

d(1 + *x/n*)/d*x*= 1/*n*. We then have

$$\begin{aligned} \frac{d(1+\frac{x}{n})^n}{dx} &= [n(1+\frac{x}{n})^{n-1}] \cdot \frac{1}{n} \\ &= (1+\frac{x}{n})^{n-1} \end{aligned}$$

But evaluating this at *x*= 0 simply gives 1, so at *x*= 0, the approximation to the derivative is exactly 1 for all values of *n*— it's not even necessary to imagine going to larger and larger values of *n*. This establishes that *c*= 1, so we have for all values of *x*.

$$\frac{de^x}{dx} = e^x$$

# Proofs of the generalizations of l'Hoˆ pital's rule

## Multiple applications of the rule



Here we prove, as claimed, that the form of Generalizations of l'Hoˆpital's rule (Page 78) can be generalized to the case where more than one application of the rule is required. The proof requires material from (Integration (Page 83)), and, as discussed in Example 86 (Page 137) , the motivation for the result becomes much more transparent once has read and knows about Sequences and Series (Page 127). The reader who has arrived here while reading Limits and continuity (Page 62) will need to defer reading this section of the proof until after Integration (Page 83), and may wish to wait until after Sequences and Series (Page 127).

The proof can be broken down into two steps.

Step 1: We first have to establish a stronger form of l'Hˆopital's rule that states that lim *u/v*= lim $\dot{u}$ / $\dot{v}$ rather than lim *u/v* = $\dot{u}$ / $\dot{v}$ . This form is stronger, because in a case like Example (Page 79) , $\dot{u}$ / $\dot{v}$ isn't defined, but lim $\dot{u}$ / $\dot{v}$ is.

We prove the stronger form using the The mean value theorem (Page 90). For simplicity of notation, let's assume that the limit is being taken at *x*= 0.

By the fundamental theorem of calculus, we have *u*(*x*) = R *x* $\dot{u}$ (*x0*) d*x0*
$u(x) = \int_0^x \dot{u}(x')dx'$ , and the mean value theorem then tells us that for some *p*between 0 and *x*, *u*(*x*) = *x* $\dot{u}$ (*p*). Likewise for a *q*in this interval, *v*(*x*) = *x*$\dot{v}$ (*q*). So

$$\lim_{x\to 0}\frac{u}{v} = \lim_{\substack{p\to 0\\ q\to 0}}\frac{\dot{u}(p)}{\dot{v}(q)}$$

but since both *p*and *q* are closer to zero than *x*is, the limit as they simultaneously approach zero is the same as the limit as *x*approaches zero.

Step 2: If we need to take *n*derivatives, the proof follows by applying the extra-strength rule *n*times. [3]

## Change of variable



We will build up the rest of the features of l'Hˆopital's rule using the technique of a change of variable. To demonstrate how this works, let's imagine that we were starting from an even more stripped-down version of l'Hˆopital's rule than the one on p. 61. Say we only knew how to do limits of the form *x*→0 rather than *x*→*a*for an arbitrary real number *a*. We could then evaluate $lim_{x\to a} u/v$ simply by defining *t* = *x*−*a* and reexpressing *u* and *v* in terms of *t*.

3. There is a logical subtlety here, which is that although we've given a clearcut recipe for cooking up a proof for any given n, that isn't quite the same thing as proving it for any positive integer n. This is an example where what we really need is a technique called proof by induction. In general, proof by induction works like this. Suppose we prove some statement about the integer 1, e.g., that l'Hˆopital's rule is valid when you take 1 derivative. Now say that we can also prove that if that statement holds for a given n, it also holds for n+ 1. Proof by induction means that we can then consider the statement as having been proved for all positive integers. For suppose the contrary. Then there would be some least nfor which it failed, but this would be a contradiction, since it would hold for n−1.

## Example 104

Reduce

$$\lim_{x \to \pi} \frac{sin x}{x - \pi}$$

to a form involving a limit at 0.

x→π x−π

.Define $t = x - \pi$. Solving for xgives $x = t + \pi$. We substitute into the above expression to find

$$\lim_{x \to \pi} \frac{sin x}{x - \pi} = \lim_{t \to 0} \frac{sin\,(t + \pi)}{t}$$

If all we knew was the →0 form of l'Hoˆ pital's rule, then this would suffice to reduce the problem to one we knew how to solve. In fact, this kind of change of variable works in all cases, not just for a limit at π, so rather then going through a laborious change of variable every time, we could simply establish the more general form on p. 61, with →*a*.

## The indeterminate form ∞/∞



To prove that l'Hˆopital's rule works in general for ∞/∞forms, we do a change of variable on the *outputs* of the functions *u*and *v*rather than their inputs. Suppose that our original problem is of the form

$$\lim \frac{u}{v}$$

where both functions blow up. [4] We then de ne U = 1=u and V = 1=v. We now have

$$\lim \frac{u}{v} = \lim \frac{1/u}{1/v} = \frac{v}{u},$$

and since *U*and *V* both approach zero, we have reduced the problem to one that can be solved using the version of l'Hˆopital's rule already proved for indeterminate forms like 0/0. Differentiating and applying the chain rule, we have

$$\lim \frac{u}{v} = \lim \frac{\dot{V}}{\dot{U}} = \lim \frac{-v^{-2}\dot{v}}{-u^{-2}\dot{u}}$$

Since lim *ab*= lim *a* lim *b* provided that lim *a*and lim *b*are both defined, we can rearrange factors to produce the desired result.

This change of variable is a specific example of a much more general method of problem-solving in which we look for a way to reduce a hard problem to an easier

4. Think about what happens when only ublows up, or only v.

one. We will encounter changes of variable again on p. 87 as a technique for integration, which means undoing the operation of differentiation.

# Proof of the fundamental theorem of calculus



There are three parts to the proof: (1) Take the equation that states the fundamental theorem, differentiate both sides with respect to *b*, and show that they're equal. (2) Show that continuous functions with equal derivatives must be essentially the same function, except for an additive constant. (3) Show that the constant in question is zero.

1. By the definition of the indefinite integral, the derivative of *x*(*b*)−*x*(*a*) with respect to *b*equals $\dot{x}$ (*b*). We have to establish that this equals the following:

$$\begin{aligned} \frac{d}{db}\int_a^b \dot{x}(t)dt &= st\frac{1}{db}\left[\int_a^{b+db}\dot{x}(t)dt - \int_a^b \dot{x}(t)dt\right] \\ &= st\frac{1}{db}\int_b^{b+db}\dot{x}(t)dt \\ &= st\frac{1}{db}\lim_{H\to\infty}\sum_{i=o}^{H}\dot{x}(b+idb/H)\frac{db}{H} \\ &= st\lim_{H\to\infty}\frac{1}{H}\sum_{i=o}^{H}\dot{x}(b+idb/H) \end{aligned}$$

Since $\dot{x}$ is continuous, all the values of $\dot{x}$ occurring inside the sum can differ only infinitesimally from $\dot{x}$ (*b*). Therefore the quantity inside the limit differs only infinitesimally from $\dot{x}$ (*b*), and the standard part of its limit must be $\dot{x}$ (*b*). [5]

2. Suppose *f* and *g* are two continuous functions whose derivatives are equal. Then *d*= *f*−*g* is a continuous function whose derivative is zero. But the only continuous function with a derivative of zero is a constant, so *f* and *g* differ by at most an additive constant.

3. I've established that the derivatives with respect to *b*of *x*(*b*) −*x*(*a*) and $\int_a^b \dot{x}dt$ are the same, so they differ by at most an additive constant.

But at *b*= *a*, they're both zero, so the constant must be zero.

5. If you don't want to use infinitesimals, then you can express the derivative as a limit, and in the final step of the argument use the mean value theorem, introduced later in the chapter.

# The intermediate value theorem



I asserted that the The intermediate value theorem (Page 63) was really more a statement about the (real or hyperreal) number system than about functions. For insight, consider figure b, which is a geometrical construction that constitutes the proof of the very first proposition in Euclid's celebrated *Elements*. The proposition to be proved is that given a line segment AB, it is possible to construct an equilateral triangle with AB as its base. The proof is by construction; that is, Euclid doesn't just give a logical argument that convinces us the triangle must exist, he actually demonstrates how to construct it. First we draw a circle with center A and radius AB, which his third postulate says we can do. Then we draw another circle with the same radius, but centered at B. Pick one of the intersections of the circles and call it C. Construct the line segments AC and BC (postulate 1). Then AC equals AB by the definition of the circle, and likewise BC equals AB. Euclid also has an axiom that things equal to the same thing are equal to one another, so it follows that AC equals BC, and therefore the triangle is equilateral.

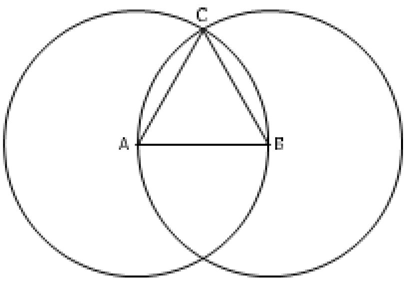


**Fig. 9.11: b / A proof from Euclid's Elements.**

It seems like a model of mathematical rigor, but there's a flaw in the reasoning, which is that he assumes without justififcation that the circles do have a point in common. To see that this is not as secure an assumption as it seems, consider the usual Cartesian representation of plane geometry in terms of coordinates ($x$, $y$). Usually we assume that $x$ and $y$are real numbers. What if we instead do our Cartesian geometry using rational numbers as coordinates? Euclid's five postulates are all consistent with this. For example, circles do exist. Let A = (0, 0) and B = (1, 0). Then there are infinitely many pairs of rational numbers in the set that satisfies the definition of the circle centered at A. Examples include (3/5, 4/5) and (−7/25, 24/25). The circle is also continuous in the sense that if I specify a point on it such as (−7/25, 24/25), and a distance that I'm allowed to make as small as I please, say $10^{-6}$, then other points exist on the circle within that distance of the given point. However, the intersection assumed by Euclid's proof doesn't exist. It would lie at (1/2, $\sqrt{3}/2$), but $\sqrt{3}$ doesn't exist in the rational number system.

In exactly the same way, we can construct counterexamples to the intermediate value theorem if the underlying system of numbers doesn't have the same properties as the real numbers. For example, let $y$ = $x^2$. Then $y$ is a continuous function, on the interval from 0 to 1, but if we take the rational numbers as our foundation, then there is no $x$ for √ which $y$ = 1/2. The solution would be $x$ = 1$\sqrt{2}$, which doesn't exist in the rational number system. Notice the similarity between this problem and the one in Euclid's proof. In both cases we have curves that cut one another without having an intersection. In the present example, the curves are the graphs of the functions *y* = $x^2$ and *y* = 1/2.

The interpretation is that the real numbers are in some sense more densely packed than the rationals, and with two thousand years worth of hindsight, we can see that Euclid should have included a sixth postulate that expressed this density property. One possible way of stating such a postulate is the following. Let L be a ray, and O its endpoint. We think of O as the origin of the positive number line. Let P and Q be sets of points on L such that every point in P is closer to O than every point in Q. Then there exists some point Z on L such that Z lies at least as far from O as every point in P, but no farther than any point in Q. Technically this property is known as *completeness*. As an example, let P = *{x* | $x^2$ <2*}* and Q = *{x* | $x^2$ ≥2*}*. Then the point Z would have to be $\sqrt{2}$, which shows that the rationals are not complete. The reals are complete, and the completeness axiom can serve as one of the fundamental axioms of the real numbers.

Note that the axiom refers to *sets* P and Q, and says that a certain fact is true for any choice of those sets; it therefore isn't the type of proposition that is covered by the transfer principle, and in fact it fails for the hyperreals, as we can see if P is the set of all infinitesimals and Q the positive real numbers.

Here is a skeletal proof of the intermediate value theorem, in which I'll make some simplifying assumptions and leave out some cases. We want to prove that if *y* is a continuous real-valued function on the real interval from *a* to *b*, and if *y*takes on values $y_1$ and $y_2$ at certain points within this interval, then for any $y_3$ between $y_1$ and $y_2$ , there is some real *x* in the interval for which *y*(*x*) = $y_3$ . I'll assume the case in which $x^1$ < $x^2$ and $y_1$ <$y_2$ .Define sets of real numbers P = *{x* |*y*≤ $y_3$ *}*, and let Q = *{x* |*y*≥ $y_3$ *}*. For simplicity, I'll assume that every member of P is less than or equal to every member of Q, which happens, for example, if the function *y*(*x*) is always increasing on the interval [*a*, *b*]. If P and Q intersect, then the theorem holds. Suppose instead that P and Q do not intersect. Using the completeness axiom, there exists some real *x* which is greater than or equal to every element of P and less than or equal to every element of Q. Suppose *x*belongs to P. Then the following statement is in the right form for the transfer principle to apply to it: for any number *x'*>*x*, *y*(*x'*) >$y_3$ . We can conclude that the statement is also true for the hyperreals, so that if d*x*is a positive infinitesimal and *x'*= *x*+ d*x*, we have *y*(*x*) < $y_3$ , but *y*(*x*+ d*x*) > $y_3$ . Then by continuity, *y*(*x*) −*y*(*x*+ d*x*) is infinitesimal. But *y*(*x*) < $y_3$ and *y*(*x*+ d*x*) > $y_3$ , so the standard part of *y*(*x*) must equal $y_3$ . By assumption *y*takes on real values for real arguments, so *y*(*x*) = $y_3$ . The same reasoning applies if *x*belongs to Q, and since *x*must belong either to P or to Q, the result is proved.

For an alternative proof of the intermediate value theorem by an entirely different technique, see Keisler (References (Page 219)).

As a side issue, we could ask whether there is anything like the interme- diate value theorem that can be applied to functions on the hyperreals. Our definition of Continuity (Page 62) explicitly states that it only applies to real functions. Even if we could apply the definition to a function on the hyperreals, the proof given above would fail, since the hyperreals lack the completeness property. As a counterexample, let be some positive infinitesimal, and define a function *y* such that *y* = −when st(*x*) ≤0 and *y* = everywhere else. If we insist on applying the definition of continuity to this function, it appears to be continuous, so it violates the intermediate value theorem. Note, however, that the way this function is defined is different from the way we usually define functions on the hyperreals. Usually we define a function on the reals, say *y* = $x^2$ , in language to which the transfer principle applies, and then we use the transfer principle to reason about the function's analog on the hyperreals. For instance, the function *y* = $x^2$ has the property that *y*≥0 everywhere, and the transfer principle guarantees that that's also true if we take *y* = $x^2$ as the definition of a function on the hyperreals.

For functions defined in this way, the intermediate value theorem makes a statement that the transfer principle applies to, and it is therefore true for the hyperreal version of the function as well.

# Proof of the extreme value theorem



The The extreme value theorem (Page 67) was stated. Before we can prove it, we need to establish some preliminaries, which turn out to be interesting for their own sake.

Definition: Let *C* be a subset of the real numbers whose definition can be expressed in the type of language to which the transfer principle applies. Then *C* is *compact* if for every hyperreal number *x* satisfying the definition of *C*, the standard part of *x* exists and is a member of *C*.

To understand the content of this definition, we need to look at the two ways in which a set could fail to satisfy it.

First, suppose *U* is defined by *x*≥0. Then there are positive infinite hyperreal numbers that satisfy the definition, and their standard part is not defined, so *U* is not compact. The reason *U* is not compact is that it is unbounded.

Second, let *V* be defined by 0 ≤*x*<1. Then if *dx*is a positive infinites-imal, 1 −*dx* satisfies the definition of *V*, but its standard part is 1, which is not in *V*, so *V*is not compact. The set *V*has boundary points at 0 and 1, and the reason it is not compact is that it doesn't contain its right-hand boundary point. A boundary point is a real number which is infinitesimally close to some points inside the set, and also to some other points that are on the outside.

We therefore arrive at the following alternative characterization of the notion of a compact set, whose proof is straightforward.

Theorem: A set is compact if and only if it is bounded and contains all of its boundary points.

Intuitively, the reason compact sets are interesting is that if you're standing inside a compact set and start taking steps in a certain direction, without ever turning around, you're guaranteed to approach some point in the set as a limit. (You might step over some gaps that aren't included in the set.) If the set was unbounded, you could just walk forever at a constant speed. If the set didn't contain its boundary point, then you could asymptotically approach the boundary, but the goal you were approaching wouldn't be a member of the set.

The following theorem turns out to be the most difficult part of the discussion.

Theorem: A compact set contains its maximum and minimum.

Proof: Let $C$be a compact set. We know it's bounded, so let $M$be the set of all real numbers that are greater than any member of $C$. By the completeness property of the real numbers, there is some real number $x$ between $C$and $M$. Let $?C$be the set of hyperreal numbers that satisfies the same definition that $C$does.

Every real $x'$ greater than $x$ fails to satisfy the condition that defines $C$, and by the transfer principle the same must be true if $x'$ is any hyperreal, so if $dx$ is a positive infinitesimal, $x+ dx$ must be outside of$?C$.

But now consider $x-dx$. The following statement holds for the reals: there is no number $x' <x$ that is greater than every member of $C$. By the transfer principle, we find that there is some hyperreal number $q$ in $?C$ that is greater than $x-dx$. But the standard part of $q$ must equal $x$, for otherwise st$q$ would be a member of $C$that was greater than $x$. Therefore $x$is a boundary point of $C$, and since $C$ is compact, $x$ is a member of $C$. We conclude $C$contains its maximum. A similar argument shows that $C$ contains its minimum, so the theorem is proved.

There were two subtle things about this proof. The first was that we ended up constructing the set of hyperreals $?C$, which was the hyperreal "big brother" of the real set $C$. This is exactly the sort of thing that the transfer principle does *not*guarantee we can do. However, if you look back through the proof, you can see that $?C$ is used only as a notational convenience. Rather than talking about whether a certain number was a member of $?C$, we could have referred, more cumbersomely, to whether or not it satisfied the condition that had originally been used to define $C$. The price we paid for this was a slight loss of generality. There are so many different sets of real numbers that they can't possibly all have explicit definitions that can be written down on a piece of paper. However, there is very little reason to be interested in studying the properties of a set that we were never able to define in the first place. The other subtlety was that we had to construct the auxiliary point $x-dx$, but there was not much we could actually say about $x-dx$ itself. In particular, it might or might not have been a member of $C$.

For example, if $C$ is defined by the condition $x= 0$, then $?C$ likewise contains only the single element 0, and $x-dx$is not a member of $?C$. But if $C$ is defined by $0 \le x \le 1$, then $x-dx$ is a member of $?C$.

The original goal was to prove the extreme value theorem, which is a statement about continuous functions, but so far we haven't said anything about functions.

Lemma: Let $f$ be a real function defined on a set of points $C$. Let $D$ be the image of $C$, i.e., the set of all values $f(x)$ that occur for some $x$in $C$. Then if $f$is continous and $C$ is compact, $D$ is compact as well. In other words, continuous functions take compact sets to compact sets. Proof: Let $y= f(x)$ be any hyperreal output corresponding to a hyperreal input *x*in *?C*. We need to prove that the standard part of *y* exists, and is a member of *D*. Since *C* is compact, the standard part of *x* exists and is a member of *C*. But then by continuity *y* differs only infinitesimally from *f*(st*x*), which is real, so st*y*= *f*(st*x*) is defined and is a member of *D*.

We are now ready to prove the extreme value theorem, in a version slightly more general than the one originally given on page 56.

The extreme value theorem: Any continuous function on a compact set achieves a maximum and minimum value, and does so at specific points in the set.

Proof: Let *f* be continuous, and let *C* be the compact set on which we seek its maximum and minimum. Then the image *D* as defined in the lemma above is compact. Therefore *D* contains its maximum and minimum values.

## Proof of the mean value theorem



Suppose that the mean value theorem is violated. Let *L*be the set of all *x*in the interval from *a*to *b*such that *y*(*x*) < $\bar{y}$, and likewise let *M*be the set with *y*(*x*) > $\bar{y}$ . If the theorem is violated, then the union of these two sets covers the entire interval from *a*to *b*. Neither one can be empty; if, for example, *M* was empty, then we would have *y*<$\bar{y}$ everywhere and also $\int_a^b y = \int_a^b \bar{y}$, but it follows directly from the definition of the definite integral that when one function is less than another, its integral is also less than the other's. Since *y* takes on values less than and greater than $\bar{y}$ , it follows from the intermediate value theorem that *y* takes on the value $\bar{y}$ somewhere (intuitively, at a boundary between *L* and *M*).

## Proof of the fundamental theorem of algebra



We start with the following lemma, which is intuitively obvious, because polynomials don't have asymptotes. Its proof is given after the proof of the main theorem.

Lemma: For any polynomial *P*(*z*) in the complex plane, its magnitude |*P*(*z*)| achieves its minimum value at some specific point $z_0$ .

The fundamental theorem of algebra: In the complex number system, a nonzero nth-order polynomial has exactly *n*roots, i.e., it can be factored into the form *P*(*z*) = (*z*−$a_1$ )(*z*− $a_2$) ...(*z*− $a_n$), where the *ai* are complex numbers.

Proof: The proofs in the cases of *n*= 0 and 1 are trivial, so our strategy is to reduce higher-*n*cases to lower ones. If an nth-degree polynomial *P* has at least one root, *a*, then we can always reduce it to a polynomial of degree *n*−1 by dividing it by (*z*−*a*). Therefore the theorem is proved by induction provided that we can show that every polynomial of degree greater than zero has at least one root.

Suppose, on the contrary, that there is an nth order polynomial *P*(*z*), with *n*>0, that has no roots at all. Then by the lemma |*P*| achieves its minimum value at some point $z_0$. To make things more simple and concrete, we can construct another polynomial *Q*(*z*) = *P*(*z*+ $z_0$)/*P*($z_0$), so that |*Q*| has a minimum value of 1, achieved at *Q*(0) = 1. This means that *Q*'s constant term is 1. What about its other terms? Let *Q*(*z*) = 1+ $c_1 z$+ ...+ $c_n z^n$. Suppose $c_1$ was nonzero. Then for infinitesimally small values of *z*, the terms of order *z*2 and higher would be negligible, and we could make *Q*(*z*) be a real number less than one by an appropriate choice of *z*'s argument. Therefore *c*1 must be zero. But that means that if $c_2$ is nonzero, then for infinitesimally small *z*, the $z^2$ term dominates the $z^3$ and higher terms, and again this would allow us to make *Q*(*z*) be real and less than one for appropriately chosen values of *z*. Continuing this process, we find that *Q*(*z*) has no terms at all beyond the constant term, i.e., *Q*(*z*) = 1. This contradicts the assumption that *n*was greater than zero, so we've proved by contradiction that there is no *P* with the properties claimed.

Uninteresting proof of the lemma: Let *M*(*r*) be the minimum value of |*P*(*z*)|on the disk defined by |*z*|≤*r*. We first prove that *M*(*r*) can't asymptotically approach a minimum as *r*approaches infinity. Suppose to the contrary: for every *r*, there is some *r*'>*r*with *M*(*r*') <*M*(*r*).

Then by the transfer principle, the same would have to be true for hyperreal values of *r*. But it's clear that if *r*is infinite, the lower-order terms of *P*will be infinitesimally small compared to the highest-order term, and therefore *M*(*r*) is infinite for infinite values of *r*, which is a contradiction, since by construction *M* is decreasing, and finite for finite *r*. We can therefore conclude by the extreme value theorem that *M* achieves its minimum for some specific value of *r*. The least such *r*describes a circle |*z*|= *r*in the complex plane, and the minimum of |*P*|on this circle must be the same as its global minimum. Applying the extreme value function to |*P*(*z*)|as a function of arg *z*on the interval 0 ≤arg*z*≤2*π*, we establish the desired result.

# Answers and solutions

## Answers to Self-Checks

### Answers to self-checks for chapter 4



Self-Check. (Page 94):

The area under the curve from 130 to 135 cm is about 3/4 of a rectangle. The area from 135 to 140 cm is about 1.5 rectangles. The number of people in the second range is about twice as much. We could have converted these to actual probabilities (1 rectangle = 5 cm ×0.005 $cm^{-1}$ = 0.025), but that would have been pointless, because we were just going to compare the two areas.

### Answers to self-checks for chapter 6



Self-Check (Page 147):

Say we're looking for *u*= √*z*, i.e., we want a number *u*that, multiplied by itself, equals *z*. Multiplication multiplies the magnitudes, so the magnitude of *u*can be found by taking the square root of the magnitude of *z*. Since multiplication also adds the arguments of the numbers, squaring a number doubles its argument. Therefore we can simply divide the argument of *z*by two to find the argument of *u*. This results in one of the square roots of *z*. There is another one, which is −*u*, since $(-u)^2$ is the same as $u^2$. This may seem a little odd:

if *u* was chosen so that doubling its argument gave the argument of *z*, then how can the same be true for −*u*? Well for example, suppose the argument of *z*is 4 ◦. Then arg *u*= 2 ◦, and arg(−*u*) = 182 ◦. Doubling 182 gives 364, which is actually a synonym for 4 degrees.

## Solutions to homework problems

### Solutions for chapter 1



Problem 1.1 (Page 20):

The tangent line has to pass through the point (3,9), and it also seems, at least approximately, to pass through (1.5,0). This gives it a slope of (9 −0)/(3 −1.5) = 9/1.5 = 6, and that's exactly what $2t$ is at $t$= 3.

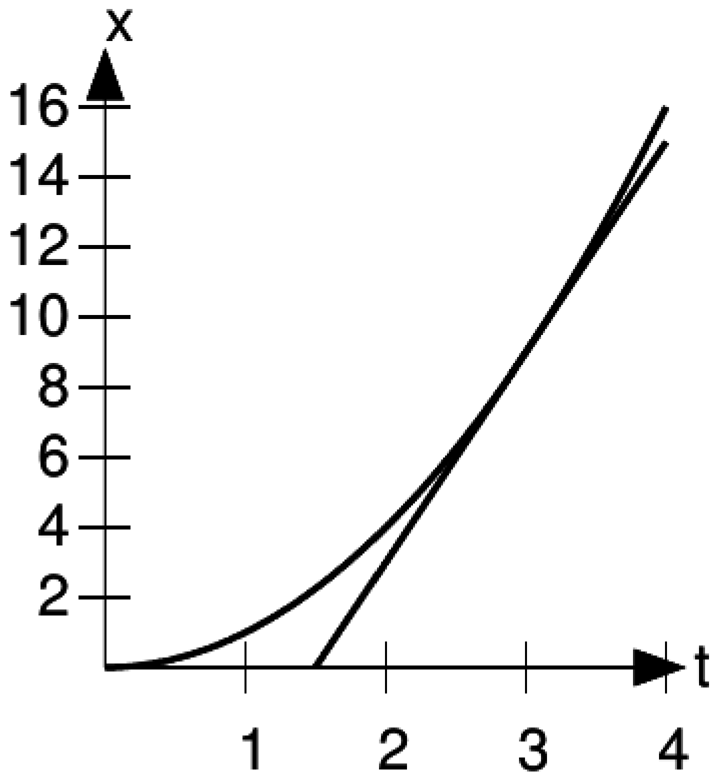


**Fig. 9.12:**

Problem 1.2 (Page 20):

The tangent line has to pass through the point (0, $\sin(e^0)$) = (0, 0.84), and it also seems, at least approximately, to pass through (-1.6,0). This gives it a slope of (0.84 −0)/(0 −(−1.6)) = 0.84/1.6 = 0.53. The more accurate result given in the problem can be found using the methods of To infinity — and beyond! (Page 25).

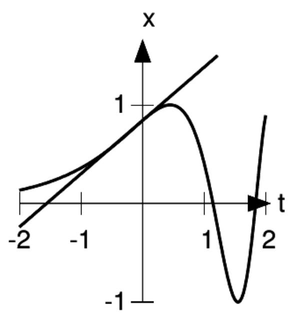


**Fig. 9.13: b / Problem 2.**

Problem 1.3 (Page 20):

The derivative is a rate of change, so the derivatives of the constants

1 and 7, which don't change, are clearly zero. The derivative can be interpreted geometrically as the slope of the tangent line, and since the functions *t*and 7*tare*lines, their derivatives are simply their slopes, 1, and 7. All of these could also have been found using the formula that says the derivative of $t^k$ is $kt^{k-1}$, but it wasn't really necessary to get that fancy. To find the derivative of $t^2$, we can use the formula, which gives 2*t*. One of the properties of the derivative is that multiplying a function by a constant multiplies its derivative by the same constant, so the derivative of 7$t^2$ must be (7)(2*t*) = 14*t*. By similar reasoning, the derivatives of $t^3$ and 7$t^3$ are 3 $t^2$ and 21 $t^2$, respectively.

Problem 1.4 (Page 20):

One of the properties of the derivative is that the derivative of a sum is the sum of the derivatives, so we can get this by adding up the derivatives of 3 $t^7$, −4$t^2$, and 6. The derivatives of the three terms are 21$t^6$, −8*t*, and 0, so the derivative of the whole thing is 21 $t^6$−8*t*.

Problem 1.5 (Page 21):

This is exactly like problem 4, except that instead of explicit numerical constants like 3 and −4, this problem involves symbolic constants *a*, *b*, and *c*. The result is 2*at*+ *b*.

Problem 1.6 (Page 21):

The first thing that comes to mind is 3*t*. Its graph would be a line with a slope of 3, passing through the origin. Any other line with a slope of3 would work too, e.g., 3*t*+ 1.

Problem 1.7 (Page 21):

Differentiation lowers the power of a monomial by one, so to get something with an exponent of 7, we need to differentiate something with an exponent of 8. The derivative of $t^8$ would be 8 $t^7$, which is eight times too big, so we really need ($t^8$/8). As in problem 6, any other function that differed by an additive constant would also work, e.g., ( $t^8$/ 8) + 1.

Problem 1.8 (Page 21):

This is just like problem 7, but we need something whose derivative is three times bigger. Since multiplying by a constant multiplies the derivative by the same constant, the way to accomplish this is to take the answer to problem 7, and multiply by three. A possible answer is (3⁄8) $t^8$, or that function plus any constant.

Problem 1.9 (Page 21):

This is just a slight generalization of problem 8. Since the derivative of a sum is the sum of the derivatives, we just need to handle each term individually, and then add up the results. The answer is (3⁄8) $t^8$ −(4⁄3) $t^3$ + 6*t*, or that function plus any constant.

Problem 1.10 (Page 22):

The function *v* = (4⁄3)*π* $(ct)^3$ looks scary and complicated, but it's nothing more than a constant multiplied by $t^3$, if we rewrite it as *v*= (4⁄3)*π* $c^3$ $t^3$. The whole thing in square brackets is simply one big constant, which just comes along for the ride when we differentiate. The result is $\dot{v}$ = (4⁄3)*π*$c^3$ (3 $t^2$), or, simplifying, $\dot{v}$ = 4*π*$c^3$ $t^2$. (For further physical insight, we can factor this as $4\pi(ct)^2 c$, where *ct* is the radius of the expanding sphere, and the part in brackets is the sphere's surface area.)

For purposes of checking the units, we can ignore the unit- less constant $4\pi$, which just leaves $c^3 t^2$. This has units of

$$(meters\, per\, second)^3$$

$(seconds)^2$, which works out to be cubic meters per second. That makes sense, because it tells us how quickly a volume is increasing over time.

Problem 1.11 (Page 22):

This is similar to problem 10, in that it looks scary, but we can rewrite it as a simple monomial,$K = (1/2)mv^2 = (1/2)m(at)^2 = (ma^2 2)t^2$. The derivative is $(ma^2/2)(2t) = ma^2 t$. The car needs more and more power to accelerate as its speed increases.

To check the units, we just need to show that the expression $ma^2 t$ has units that are like those of the original expression for *K*, but divided by seconds, since it's a rate of change of *K* over time. This indeed works out, since the only change in the factors that aren't unitless is the reduction of the powet of *t* from 2 to 1.

Problem 1.12 (Page 22):

The area is $a = l^2 = (1 + \alpha T)^2 l_0^2$To make this into something we know how to differentiate, we need to square out the expression involving *T*, and make it into something that is expressed explicitly as a polynomial:

$$a = l_0^2 + 2l_0^2 \alpha T + l_0^2 \alpha^2 T^2$$

Now this is just like problem 5, except that the constants superficially look more complicated. The result is

$$\dot{a} = 2_0^2 \alpha + 2l_0^2 \alpha^2 T$$

$$= 2_0^2(\alpha + l_0^2 \alpha^2)$$

We expect the units of the result to be area per unit temperature, e.g., degrees per square meter. This is a little tricky, because we have to figure out what units are implied for the constant $\alpha$. Since the question talks about 1 + $\alpha T$, apparently the quantity $\alpha T$is unitless. (The 1 is unitless, and you can't add things that have different units.) Therefore the units of $\alpha$ must be "per degree," or inverse degrees. It wouldn't make sense to add $\alpha$ and $\alpha^2 T$ unless they had the same units (and you can check for yourself that they do), so the whole thing inside the parentheses must have units of inverse degrees. Multiplying by the $l_0^2$ in front, we have units of area per degree, which is what we expected.

Problem 1.13 (Page 22):

The first derivative is $6t^2 - 1$. Going again, the answer is 12*t*.

Problem 1.14 (Page 23):

The first derivative is $3t^2 + 2t$, and the second is $6t+2$. Setting this equal to zero and solving for *t*, we find $t = -1/3$. Looking at the graph, it does look like the concavity is down for $t < -1/3$, and up for $t > -1/3$.

Problem 1.15 (Page 23):

I chose $k = -1$, and $t = 1$. In other words, I'm going to check the slope of the function $x = t^{-1} = 1/r$ at $t = 1$, and see whether it really equals

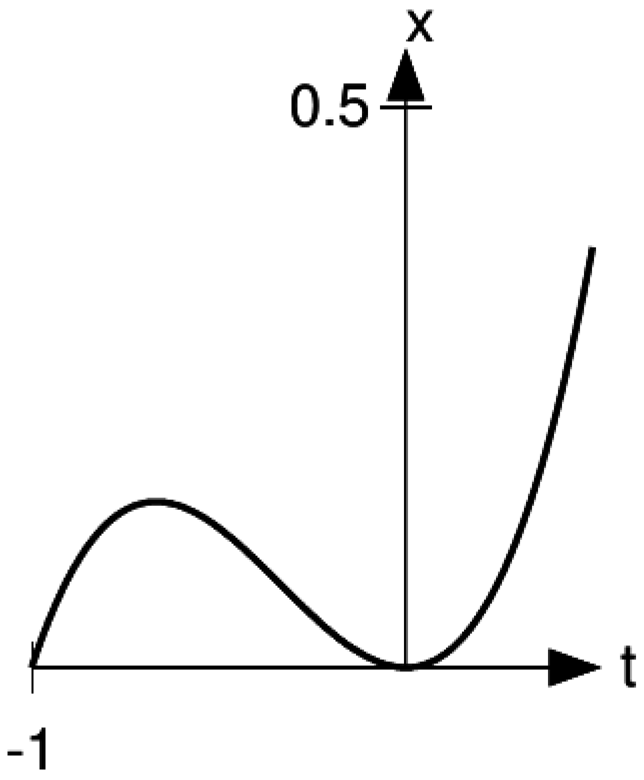


**Fig. 9.14: c /**

$kt^{k-1} = -1$. Before even doing the graph, I note that the sign makes sense: the function $1/t$is decreasing for $t>0$, so its slope should indeed be negative.

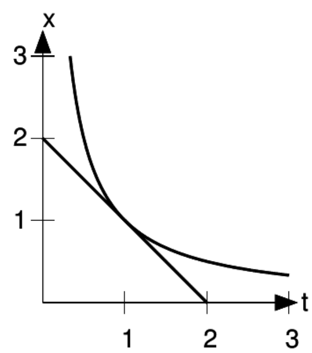


**Fig. 9.15: d / Problem 15.**

The tangent line seems to connect the points (0,2) and (2,0), so its slope does indeed look like it's −1.

The problem asked us to consider the logical meaning of the two possible outcomes. If the slope had been significantly different from−1 given the accuracy of our result, the conclusion would have been that it was incorrect to extend the rule to negative values of $k$. Although our example did come out consistent with the rule, that doesn't prove the rule in general. An example can disprove a conjecture, but can't prove it. Of course, if we tried lots and lots of examples, and they all worked, our confidence in the conjecture would be increased.

Problem 1.16 (Page 23):

A minimum would occur where the derivative was zero. First we rewrite the function in a form that we know how to differentiate:

$$E(r) = ka^{12}r^{-12} - 2ka^6r^{-6}$$

We're told to have faith that the derivative of $k^t$ is $kt^{k-1}$ even for $k$<0, so

$$\begin{aligned} 0 &= \dot{E} \\ &= -12ka^{12}r^{-13} + 12ka^6r - 7 \end{aligned}$$

To simplify, we divide both sides by 12$k$. The left side was already zero, so it keeps being zero.

$$\begin{aligned} 0 &= -a^{12}r^{-13} + a^6r^{-7} \\ a^{12}r^{-13} &= a^6r^{-7} \\ a^{12} &= a^6r^6 \\ a^6 &= r^6 \\ r &= \pm a \end{aligned}$$

To check that this is a minimum, not a maximum or a point of inflection, one method is to construct a graph. The constants $a$and $k$are irrelevant to this issue. Changing $a$just rescales the horizontal $r$axis, and changing $k$does the same for the vertical $E$axis. That means we can arbitrarily set $a$= 1 and $k$= 1, and construct the graph shown in the figure. The points $r$= ±$a$are now simply $r$= ±1. From the graph, we can see that they're clearly minima. Physically, the minimum at $r$= −$a$can be interpreted as the same physical configuration of the molecule, but with the positions of the atoms reversed. It makes sense that $r$= −$a$behaves the same as $r$= $a$, since physically the behavior of the system has to be symmetric, regardless of whether we view it from in front or from behind.

The other method of checking that $r$= $a$is a minimum is to take the second derivative. As before, the values of $a$and $k$are irrelevant, and can be set to 1. We then have

$$\begin{aligned} \dot{E} &= -12r^{-13} + 12r^{-7} \\ \ddot{E} &= -156r^{-14} - 84r^{-8} \end{aligned}$$

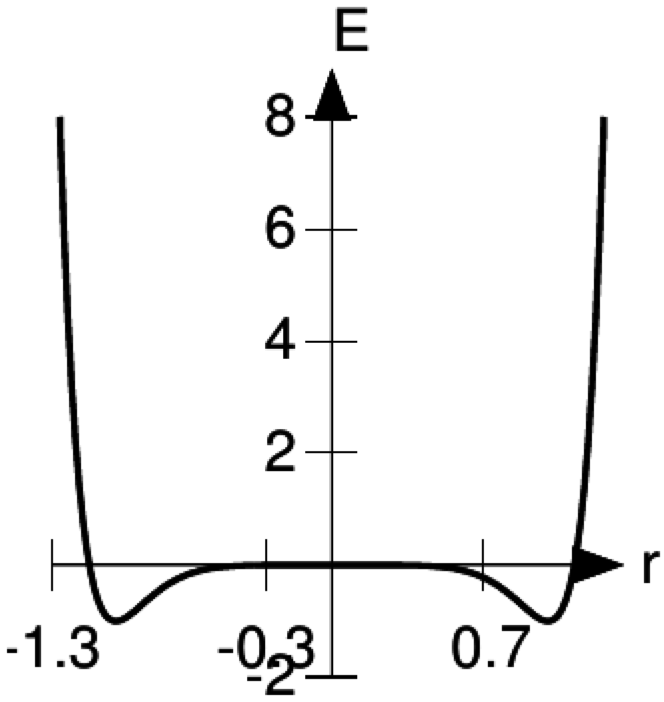


**Fig. 9.16: e / Problem 16.**

Plugging in $r$= ±1, we get a positive result, which confirms that the concavity is upward.

Problem 1.17 (Page 23):

Since polynomials don't have kinks or endpoints in their graphs, the maxima and minima must be points where the derivative is zero. Differentiation bumps down all the powers of a polynomial by one, so the derivative of a third-order polynomial is a second-order polynomial. A second-order polynomial can have at most two real roots (values of $t$for which it equals zero), which are given by the quadratic formula. (If the number inside the square root in the quadratic formula is zero or negative, there could be less than two real roots.) That means a third-order polynomial can have at most two maxima or minima.

Problem 1.18 (Page 24):

Since $f$, $g$, and $s$are smooth and defined everywhere, any extrema they possess occur at places where their derivatives are zero. The converse is not necessarily true, however; a place where the derivative is zero could be a point of inflection. The derivative is additive, so if *both* $f$ and $g$ have zero derivatives at a certain point, $s$does as well. Therefore in most cases, if $f$and $g$both have an extremum at a point, so will $s$. However, it could happen that this is only a point of inflection for $s$, so in general, we can't conclude anything about the extrema of $s$simply from knowing where the extrema of $f$ and $g$o ccur.

Going the other direction, we certainly can't infer anything about extrema of $f$ and $g$ from knowledge of $s$ alone. For example, if $s(x)$ = $x^2$ , with a minimum at $x$= 0, that tells us very little about *f* and *g*. We could have, for example, $f(x) = (x-1)^2 2 - 2$ *and* $(x+1)^2/2+1$ neither of which has an extremum at x = 0.

Problem 1.19 (Page 24):

Considering *V* as a function of *h*, with *b* treated as a constant, we have for the slope of its graph

$$\dot{V} = \frac{e_v}{e_h},$$

so

$$\begin{aligned} e_v &= \dot{V} \cdot e_h \\ &= \frac{1}{3} b e_h \end{aligned}$$

Problem 1.20 (Page 24):

Thinking of the rocket's height as a function of time, we can see that goal is to measure the function at its maximum. The derivative is zero at the maximum, so the error incurred due to timing is approximately zero. She should not worry about the timing error too much. Other factors are likely to be more important, e.g., the rocket may not rise exactly vertically above the launch pad.

Problem 1.21 (Page 24):

If $\dot{x} = n^2$, and *x* is a polynomial in *n*, then

we must have $\dot{x}$ (*n*) = *x*(*n*) −*x*(*n*−1) = $n^2$. If *x* is a polynomial of order *k*, then *x*(*n*) and *x*(*n*−1) both have $n^k$ terms with coefficients of 1, so $\dot{x}$ has no $n^k$ term. We want *x* ˙ to have a nonvanishing $n^2$ term, so we must have *k*≥3. For *k*>3, it's easy to show that the $n^3$ term in *x*(*n*) −*x*(*n*−1) is nonzero, so we must have *k*= 3. Let *x*(*n*) = *a* $n^3$ + *b* $n^2$ + ..., where *a* is the coefficient that we want to prove is 1/3, and ... represents lower-order terms. By the binomial theorem, we have *x*(*n*−1) = *an*3 −3*a* $n^2$ + *b* $n^2$ + ..., and subtracting this from *x*(*n*) gives $\dot{x}$ (*n*) = 3*a* $n^3$ + .... Since 3*a*= 1, we have *a*= 1/3.

## Solutions for chapter 2



Problem 2.1 (Page 51):

$$\begin{aligned} \frac{dx}{dt} &= \frac{(t+dt)^4 - t^4}{dt} \\ &= \frac{a4t^3dt + 6t^2dt^2 + 4tdt^3 + dt^4}{dt} \\ &= 4t^{3+\dots,} \end{aligned}$$

where . . . indicates infinitesimal terms. The derivative is the standard part of this, which is $4t^3$.

Problem 2.2 (Page 52):

$$\frac{dx}{dt} = \frac{cos(t + dt) - cost}{dt}$$

The identity cos(*α*+ *β*) = cos *α*cos *β*−sin *α*sin *β*then gives

$$\frac{dx}{dt} = \frac{costcosdt - sintsindt - cost}{dt}$$

The small-angle approximations cos d*t*≈1 and sin d*t*≈d*t*result in

$$\frac{dx}{dt} = \frac{-sintdt}{dt}$$
$$= -sint$$

Problem 2.3 (Page 52):

| H | $\sqrt{H+1} - \sqrt{H-1}$ |
|---|---|
| 1000 | .032 |
| 1000,000 | 0.0010 |
| 1000,000,000 | 0.00032 |

The result is getting smaller and smaller, so it seems reasonable to guess that if *H*is infinite, the expression gives an infinitesimal result.

Problem 2.4 (Page 52):

| dx | $\sqrt{dx}$ |
|---|---|
| .1 | .23 |
| .001 | .032 |
| .00001 | 0..32 |

The square root is getting smaller, but is not getting smaller as fast as the number itself. In proportion to the original number, the square root is actually getting bigger. It looks like $\sqrt{dx}$ is infinitesimal, but it's still infinitely big compared to dx. This makes sense, because $\sqrt{dx}$ equals $dx^{1/2}$. we already knew that $dx^0$, which equals 1, was infinitely big compared to dx^{1} , which equals dx. In the hierarchy of infinitesimals, $dx^{1/2}$ fits in between $dx^0$ and $dx^1$.

Problem 2.5 (Page 52):

Statements (a)-(d), and (f )-(g) are all valid for the hyperreals, because they meet the test of being directly translatable, without having to interpret the meaning of things like particular subsets of the reals in the context of the hyperreals.

Statement (e), however, refers to the rational numbers, a particular subset of the reals, and that means that it can't be mindlessly translated into a statement about the hyperreals, unless we had figured out a way to translate the set of rational numbers into some corresponding subset of the hyperreal numbers like the hyperrationals! This is not the type of statement that the transfer principle deals with. The statement is not true if we try to change "real" to "hyperreal" while leaving "rational" alone; for example, it's not true that there's a rational number that lies between the hyperreal numbers 0 and 0 + d*x*, where d*x*is infinitesimal.

Problem 2.6 (Page 53):

If $R_1$ is finite and $R_2$ infinite, then 1/ $R_2$ is infinitesimal, 1/$R_1$ + 1/$R_2$ differs infinitesimally from 1/$R_1$ , and the combined resistance *R* differs infinitesimally from $R_1$ . Physically, the second pipe is blocked or too thin to carry any significant flow, so it's as though it weren't present.

If $R_1$ is finite and $R_2$ is infinitesimal, then 1/$R_2$ is infinite, 1/$R_1$ + 1/$R_2$ is also infinite, and the combined resistance *R*is infinitesimal. It's so easy for water to flow through $R_2$ that $R_1$ might as well not be present. In the context of electrical circuits rather than water pipes, this is known as a short circuit.

Problem 2.7 (Page 53):

The velocity addition is only interesting if the infinitesimal velocities *u*and *v*are comparable to one another, i.e., their ratio is finite. Let's writefor the size of these infinitesimals, so that both *u*and *v*can be written asmultiplied by some finite number. Then 1 + *uv* differs from 1 by an amount that is on the order of $\varepsilon^2$, which is infinitesimally small compared to . The same then holds true for 1/(1 + *uv*) as well. The result of velocity addition (*u*+ *v*)/(1 + *uv*) is then *u*+ *v*+ ..., where ...represents quantities of order $\varepsilon^3$, which are amount to a correction that is infinitesimally small compared to the nonrelativistic result *u*+ *v*.

Problem 2.8 (Page 53):

This would be a horrible problem if we had to expand this as a polynomial with 101 terms, as in chapter 1! But now we know the chain rule, so it's easy. The derivative is

$$[100(2x+3)^{99}][2]$$

where the first factor in brackets is the derivative of the function on the outside, and the second one is the derivative of the "inside stuff." Simplifying a little, the answer is $200(2x+3)^{99}$.

Problem 2.9 (Page 54):

Applying the product rule, we get

$$(x+1)^{99}(x+2)^{200}+(x+1)^{100}(x+2)^{199}$$

(The chain rule was also required, but in a trivial way — for both of the factors, the derivative of the "inside stuff " was one.)

Problem 2.10 (Page 54):

The derivative of $e^{7x}$ is $e^{7x}\cdot 7$, where the first factor is the derivative of the outside stuff (the derivative of a base-*e*exponential is just the same thing), and the second factor is the derivative of the inside stuff. This would normally be written as $7e^{7x}$.

The derivative of the second function is $e^{e^x}e^x$, with the second exponential factor coming from the chain rule.

Problem 2.11 (Page 54):

We need to put together three different ideas here: (1) When a function to be differentiated is multiplied by a constant, the constant just comes along for the ride. (2) The derivative of the sine is the cosine. (3) We need to use the chain rule. The result is −*ab*cos(*bx*+ *c*).

Problem 2.13 (Page 54):

If we just wanted to fine the integral of sin *x*, the answer would be −cos *x*(or −cos *x*plus an arbitrary constant), since the derivative would be −(−sin *x*), which would take us back to the original function. The obvious thing to guess for the integral of *a*sin(*bx*+ *c*) would therefore be −*a*cos(*bx*+ *c*), which almost works, but not quite. The derivative of this function would be *ab*sin(*bx*+ *c*), with the pesky factor of *b*coming from the chain rule. Therefore what we really wanted was the function −(*a/b*) cos(*bx*+ *c*).

Problem 2.14 (Page 55):

The chain rule gives

$$\frac{d}{dx}((x^2)^2)^2 = 2((x^2)^2)(2(x^2))(2x) = 8x^7$$

which is the same as the result we would have gotten by differentiating $x^8$.

Problem 2.15 (Page 55):

To find a maximum, we take the derivative and set it equal to zero. The whole factor of $2v^2$/ *g*in front is just one big constant, so it comes along for the ride. To differentiate the factor of sin *θ*cos *θ*, we need to use the chain rule, plus the fact that the derivative of sin is cos, and the derivative of cos is −sin.

$$0 = \frac{2v^2}{g}(cos\theta cos\theta + sin\theta(-sin\theta))$$
$$0 = cos^2\theta - sin^2\theta$$
$$cos\theta = \pm sin\theta$$

We’re interested in angles between, 0 and 90 degrees, for which both the sine and the cosine are positive, so

$$cos\theta = sin\theta$$
$$tan\theta = 1$$
$$\theta = 45^\circ$$

To check that this is really a maximum, not a minimum or an inflection point, we could resort to the second derivative test, but we know the graph of $R(\theta)$ is zero at $\theta = 0$ and $\theta = 90$ ◦, and positive in between, so this must be a maximum.

Problem 2.17 (Page 55):

Taking the derivative and setting it equal to zero, we have $(e^x - e^{-x})/2 = 0$, so $e^x = e^{-x}$, which occurs only at $x = 0$. The second derivative is $(e^x + e^{-x})/2$ (the same as the original function), which is positive for all $x$, so the function is everywhere concave up, and this is a minimum.

Problem 2.18 (Page 55):

There are no kinks, endpoints, etc., so extrema will occur only in places where the derivative is zero. Applying the chain rule, we find the derivative to be cos(sin(sin $x$)) cos(sin $x$) cos $x$. This will be zero if any of the three factors is zero. We have cos $u = 0$ only when $|u| \geq \pi/2$, and $\pi/2$ is greater than 1, so it's not possible for either of the first two factors to equal zero. The derivative will therefore equal zero if and only if cos $x = 0$, which happens in the same places where the derivative of sin $x$is zero, at $x = \pi/2 + \pi n$, where $n$is an integer.

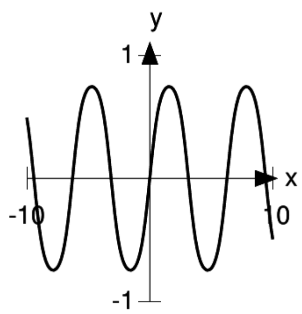


**Fig. 9.17: Problem 18.**

This essentially completes the required demonstration, but there is one more technical issue, which is that it's conceivable that some of these could be points of inflection. Constructing a graph of sin (sin (sin $x$)) gives us the necessary insight to see that this can't be the case. The function essentially looks like the sine function, but its extrema have been "shaved down" a little, giving them slightly flatter tips that don't quite extend out to ±1. It's therefore fairly clear that these aren't points of inflection. To prove this more rigorously, we could take the second derivative and show that it was nonzero at the places where the first derivative is zero. That would be messy. A less tedious argument is as follows. We can tell from its formula that the function is *periodic*, i.e., it has the property that $f(x+$ `) = $f(x)$, for `= $2\pi$. This follows because the innermost sine function is periodic, and the outer layers only depend on the result of the inner layer. Therefore all the points of the form $\pi/2 + 2\pi n$have the same behavior. Either they're all maxima or they're all points of inflection. But clearly a function can't oscillate back and forth without having any maxima at all, so they must all be maxima. A similar argument applies to the minima.

Problem 2.19 (Page 56):

The function $f$has a kink at $x = 0$, so it has no uniquely defined tangent line there, and its derivative at that point is undefined. In terms of infinitesimals, positive values of d*x*give d*f*/d*x*= (d*x*+ d*x*)/d*x*= 2, while negative ones give d*f*/d*x* = (−d*x*+ d*x*)/d*x*= 0. Since the standard part of the quotient d*y*/d*x*depends on the specific value of d*x*, the derivative is undefined.

The function *g*has no kink at *x*= 0. The graph of *x*|*x*|looks like two half-parabolas glued together, and since both of them have slopes of 0 at *x*= 0, the slope of the tangent line is well defined, and is zero. In terms of infinitesimals, d*g*/d*y*is the standard part of |d*x*|+ 1, which is 1.

Problem 2.20 (Page 56):

(a) As suggested, let $c = \sqrt{g/A}$, so that *d* = *A*ln cosh *ct*= *A* ln$(e^{ct} + e^{-ct})$. Applying the chain rule, the velocity is

$$A\frac{ce^{ct} - ce^{-ct}}{coshct}$$

(b) The expression can be rewritten as *Ac*tanh *ct*.

(c) For large *t*, the $e^{-ct}$ terms become negligible, so the velocity is $Ace^{ct}/e^{ct}$= *Ac*.
(d) From the original expression, *A*must have units of distance, since the logarithm is unitless. Also, since *ct*occurs inside a function, *ct*must be unitless, which means that *c*has units of inverse time. The answers to parts b and c get their units from the factors of *Ac*, which have units of distance multiplied by inverse time, or velocity.

Problem 2.21 (Page 56):

Since I've advocated not memorizing the quotient rule, I'll do this one from first principles, using the product rule.

$$\begin{aligned}&\frac{d}{d\theta}tan\theta\\&=\frac{d}{d\theta}(\frac{sin\theta}{cos\theta})\\&=\frac{d}{d\theta}[sin\theta(cos\theta)^{-1}]\\&=cos\theta(cos\theta)^{-1}+(sin\theta)\,(-1)(cos\theta)^{-2}(-sin\theta)\\&=1+tan^2\theta\end{aligned}$$

(Using a trig identity, this can also be rewritten as $sec^2\theta$)

Problem 2.22 (Page 57):

Reexpressing $\sqrt[3]{x}$ as $x^{1/3}$, the derivative is $(1/3)x^{-2/3})$.

Problem 2.23 (Page 57):

(a) Using the chain rule, the derivative of $(x^2+1)^{1/2}$ is (1/2) $(x^2+1)^{-1/2}(2x) = x(x^2+1)^{-1/2}$

(b) This is the same as a, except that the 1 is replaced with an *a*2 , so the answer is $x(x^2+a^2)^{-1/2}$. The idea would be that *a*has the same units as *x*.

(c) This can be rewritten as $(a + x)^{-1/2}$, giving a derivative of $(-1/2)(a + x)^{-3/2}$

(d) This is similar to c, but we pick up a factor of −2*x*from the chain rule, making the result $ax(a - x^2)^{-3/2}$.

Problem 2.24 (Page 57):

By the chain rule, the result is 2/(2*t*+ 1).

Problem 2.25 (Page 57):

Using the product rule, we have

$$(\frac{d}{dx}3)sinx + 3(\frac{d}{dx}sinx),$$

but the derivative of a constant is zero, so the first term goes away, and we get 3 cos *x*, which is what we would have had just from the usual method of treating multiplicative constants.

Problem 2.26 (Page 57):

N(Gamma(2))
*1*
N(Gamma(2.00001))
*1.0000042278*
N( (1.0000042278-1)/(.00001) )
*0.4227799998*

Probably only the first few digits of this are reliable.

Problem 2.27 (Page 58):

The area and volume are

$$A = 2\pi rl + 2\pi r^2$$

and

$$V = \pi r^2 l$$

.

The strategy is to use the equation for *A*, which is a constant, to eliminate the variable `, and then maximize *V*in terms of *r*.

$$l = A - 2\pi r^2/2\pi r$$

Substituting this expression for `back into the equation for *V*,

$$V = \frac{1}{2}rA - \pi r^3$$

To maximize this with respect to *r*, we take the derivative and set it equal to zero.

$$0 = \frac{1}{2}A - 3\pi r^2$$
$$A = 6\pi r^2$$
$$l = (6\pi r^2 - 2\pi r^2)/2\pi$$
$$rl = 2r$$

In other words, the length should be the same as the diameter.

Problem 2.28 (Page 58):

(a) We can break the expression down into three factors: the constant *m/*2 in front, the nonrelativistic velocity dependence $v^2$, and the relativistic correction factor $(1 - v^2/c^2)^{-1/2}$. Rather than substituting in *at* for *v*, it's a little less messy to calculate d*K/*d*t*= (d*K/*d*v*)(d*v/*d*t*) = *a*d*K/*d*v*. Using the product rule, we have

$$\begin{aligned}\frac{dK}{dt} &= a \cdot \frac{1}{2}m[2v(1 - \frac{v^2}{c^2})^{-1/2} \\ &+ v^2 \cdot \left(-\frac{1}{2}\right)(1 - \frac{v^2}{c^2})^{-3/2}(-\frac{2v}{c^2})] \\ &= ma^2t[(1 - \frac{v^2}{c^2})^{-1/2} + \frac{v^2}{2c^2}(1 - \frac{v^2}{2c^2})^{-3/2}]\end{aligned}$$

(b) The expression $ma^2t$ is the nonrelativistic (classical) result, and has the correct units of kinetic energy divided by time. The factor in square brackets is the relativistic correction, which is unitless.

(c) As *v*gets closer and closer to *c*, the expression 1 $-v^2/c^2$ approaches zero, so both the terms in the relativistic correction blow up to positive infinity.

Problem 2.29 (Page 58):

We already know it works for positive *x*, so we only need to check it for negative *x*. For negative values of *x*, the chain rule tells us that the derivative is 1/|*x*|, multiplied by −1, since d|*x*|/d*x*= −1. This gives −1/|*x*|, which is the same as 1/*x*, since *x*is assumed negative.

Problem 2.30 (Page 58):

Since *f*(*x*) = *f*(−*x*),

$$\frac{df(x)}{dx} = \frac{df(-x)}{dx}$$

But by the chain rule, the right-hand side equals −*f*′(*x*), as claimed.

Problem 2.32 (Page 59):

Let *f* = $dx^k$ /d*x* be the unknown function. Then

$$\begin{aligned}1 &= \frac{dx}{dx}\\ &= \frac{d}{dx}(x^k x^{-k+1})\\ &= fx^{-k+1} + x^k(-k+1)x^{-k}\end{aligned}$$

where we can use the ordinary rule for derivatives of powers on $x^{-k+1}$, since −*k*+ 1 is positive. Solving for *f*, we have the desired result.

Problem 2.33 (Page 59):

Since the parallel postulate can be expressed in terms of algebra through Cartesian geometry, the transfer principle tells us that it holds for F as well. But G is defined in terms of the finite hyperreals, so statements about E don't carry over to statements about G simply by replacing "real" with "hyperreal," and the transfer principle does not guarantee that the parallel postulate applies to G.

In fact, it is easy to find a counterexample in G. Letbe an infinitesimal number. Consider the lines with equations *y*= 1 and *y*= 1+*x*. Neither of these intersects the *x* axis.

No, it is not valid to associate only E with the plane described by Eu-clid's axioms. All of Euclid's axioms hold equally well in F. F is referred to as a nonstandard model of Euclid's axioms. It has the same relation to standard Euclidean geometry as the hyperreals have to the reals. If we want to make up a set of axioms that describes E and can't describe F, then we need to add an additional axiom to Euclid's set. An example of such an axiom would be an axiom stating that given any two line segments with lengths $l_1$ and $l_2$, there exists some integer *n*such that *n* $l_1 > l_2$. Note that although this axiom holds in E, the transfer principle cannot be used to show that it holds in F — it is false in F. The transfer principle doesn't apply because the transfer principle doesn't apply to statements that include phrases such as "for any integer."

Problem 2.34 (Page 59):

The normal definition of a repeating decimal such as 0.999 …is that it is the *limit* of the sequence 0.9, 0.99, …, and the limit is a real number, by definition. 0.999 …equals 1. However, there is an intuition that the limiting process 0.9, 0.99, …"never quite gets there." This intuition can, in fact, be formalized in the construction described beginning on page 144; we can define a hyperreal number based on the sequence 0.9, 0.99, …, and it is a number infinitesimally less than one. This is not, however, the normal way of defining the symbol 0.999 …, and we probably wouldn't want to change the definition so that it was. If it was, then 0.333 …would not equal 1⁄3.

Problem 2.35 (Page 60):

Converting these into Leibniz notation, we find

$$\frac{df}{dx} = \frac{dg}{dh}$$

and

$$\frac{df}{dx} = \frac{dg}{dh} \cdot h \cdot$$

To prove something is not true in general, it suffices to find one counterexample. Suppose that *g*and *h*are both unitless, and *x*has units of seconds. The value of *f*is defined by the output of *g*, so *f*must also be unitless. Since *f* is unitless, d*f* / d*x* has units of inverse seconds ("per second"). But this doesn't match the units of either of the proposed expressions, because they're both unitless. The correct chain rule, however, works. In the equation

$$\frac{df}{dx} = \frac{dg}{dh} \cdot \frac{dh}{dx},$$

the right-hand side consists of a unitless factor multiplied by a factor with units of inverse seconds, so its units are inverse seconds, matching the left-hand side.

Problem 2.36 (Page 60):

We can make life a lot easier by observing that the function *s*(*f*) will be maximized when the expression inside the square root is minimized. Also, since *f*is squared every time it occurs, we can change to a variable *x*= $f^2$, and then once the optimal value of *x*is found we can take its square root in order to find the optimal *f*. The function to be optimized is then

$$a(x - f_0^2)^2 + bx$$

Differentiating this and setting the derivative equal to zero, we find

$$2a(x - f_0^2) + b = 0$$

which results in $x = f_0^2 - b/2a$, or

$$f = \sqrt{f_0^2 - b/2a}$$

(choosing the positive root, since *f*represents a frequencies, and frequencies are positive by definition). Note that the quantity inside the square root involves the square of a frequency, but then we take its square root, so the units of the result turn out to be frequency, which makes sense. We can see that if *b*is small, the second term is small, and the maximum occurs very nearly at *f*o .

There is one subtle issue that was glossed over above, which is that the graph on page 51 shows *two* extrema: a minimum at *f*= 0 and a maximum at *f*>0. What happened to the *f*= 0 minimum? The issue is that I was a little sloppy with the change of variables. Let *I*stand for the quantity inside the square root in the original expression for *s*. Then by the chain rule,

$$\frac{ds}{df} = \frac{ds}{dI} \cdot \frac{dI}{dx} \cdot \frac{dx}{df}$$

We looked for the place where d*I*/d*x*was zero, but d*s*/d*f*could also be zero if one of the other factors was zero. This is what happens at *f*= 0, where d*x*/d*f*= 0.

Problem 2.37 (Page 61):

$$\begin{aligned} y &= (\frac{1}{f} - \frac{1}{x})^{-1} \\ &= (\frac{1}{f} - \frac{1}{f + dx})^{-1} \\ &= (\frac{1}{f} - \frac{1}{1 + dx/f})^{-1} \end{aligned}$$

Applying the geometric series $1/(1 + r) = 1 + r + r^2 + \ldots$,

$$\begin{aligned} y &\approx f(1 - (1 - \frac{dx}{f}))^{-1} \\ &= \frac{f^2}{dx} \end{aligned}$$

As checks on our result, we note that the units work out correctly (meters squared divided by meters give meters), and that the result is indeed large, since we divide by the small quantity d*x*.

Problem 2.38 (Page 61):

One way to evaluate an expression like $a^b$ is by using the identity $a^b = e^{b \ln a}$. If we try to substitute $a = 1$ and $b = \infty$, we get $e^{\infty - 0}$, which has an indeterminate form inside the exponential.

One way to express the idea is that if there is even the tiniest error in the value of *a*, the value of $a^\infty$ can have any positive value.

## Solutions for chapter 3



Problem 3.1 (Page 80)

(a) The Weierstrass definition requires that if we're given a particular, and we be able to find a $\delta$ so small that $f(x) + g(x)$ differs from $F + G$ by at mostfor $|x - a| < \delta$. But the Weierstrass definition also tells us that given/2, we can find a $\delta$ such that *f* differs from *F* by at most /2, and likewise for *g* and *G*. The amount by which $f + g$ differs from $F + G$ is then at most/2 +/2, which completes the proof.

(b) Let d*x* be infinitesimal. Then the definition of the limit in terms of infinitesimals says that the standard part of $f(a + dx)$ differs at most infinitesimally from *F*, and likewise for *g* and *G*. This means that $f + g$ differs from $F + G$ by the sum of two infinitesimals, which is itself an infinitesimal, and therefore the standard part of $f+g$ evaluated at $x+dx$ equals $F + G$, satisfying the definition.

Problem 3.2 (Page 80):

The shape of the graph can be found by considering four cases: large negative *x*, small negative *x*, small positive *x*, and large positive *x*. In these four cases, the function is respectively close to 1, large, small, and close to 1.

The four limits correspond to the four cases described above.

Problem 3.3 (Page 81)

All five of these can be done using l'Hˆopital's rule:

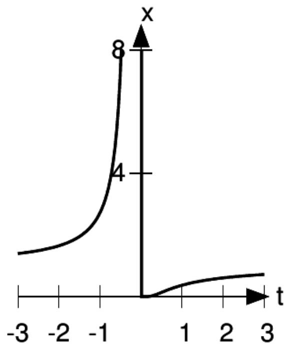


**Fig. 9.18:**

$$\lim_{s\to 1}\frac{s^3-1}{s-1}=lim\frac{3s^2}{1}=3$$

$$\lim_{\theta\to 0}\frac{1-cos\theta}{\theta^2}=lim\frac{sin\theta}{2\theta}=lim\frac{cos\theta}{2}=\frac{1}{2}$$

$$\lim_{x\to\infty}\frac{5x^2-2x}{x}=lim\frac{10x-2}{1}=\infty$$

$$\lim_{n\to\infty}\frac{n(n+1)}{(n+2)(n+3)}=lim\frac{n^2+\ldots}{n^2+\ldots}=lim\frac{2n+\ldots}{2n+\ldots}=lim\frac{2}{2}=1$$

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=lim\frac{2ax+\ldots}{2dx+\ldots}=lim\frac{2a+\ldots}{2a+\ldots}=lim\frac{2a}{2d}=\frac{a}{d}$$

In examples 2, 4, and 5, we differentiate more than once in order to get an expression that can be evaluated by substitution. In 4 and 5, . . . represents terms that we anticipate will go away after the second differentiation. Most people probably would not bother with l'Hˆopital's rule for 3, 4, or 5, being content merely to observe the behavior of the highest-order term, which makes the limiting behavior obvious. Examples 3, 4, and 5 can also be done rigorously without l'Hˆopit rule, by algebraic manipulation; we divide on the top and bottom by the highest power of the variable, giving an expression that is no longer an indeterminate form $\infty/\infty$.

Problem 3.4 (Page 81)

Both numerator and denominator go to zero, so we can apply l'Hˆopital's rule. Differentiating top and bottom gives (cos *x*−*x*sin *x*)/(−ln 2 ·$2^x$ ), which equals −1/*ln*2 at *x*= 0. To check this numerically, we plug *x*= $10^{-3}$ into the original expression. The result is −1.44219, which is very close to −1/*ln*2 = −1.44269 ....

Problem 3.5 (Page 81)

L'Hˆopital's rule only works when both the numerator and the denominator go to zero.

Problem 3.6 (Page 82)

Applying l'Hˆopital's rule once gives

$$\lim_{u \to 0} \frac{2u}{e^u + e^{-u}},$$

which is still an indeterminate form. Applying the rule a second time, we get

$$\lim_{u \to 0} \frac{2}{e^u + e^{-u}} = 1,$$

As a numerical check, plugging $u$= 0.01 into the original expression results in 0.9999917.

Problem 3.7 (Page 82)

L'Hˆopital's rule gives cos $t$/1 →−1. Plugging in $t$= 3.1 gives -0.9997.

Problem 3.8 (Page 82)

Let $u$= 1/$x$. Then

$$\frac{df/dx}{dg/dx} = \frac{df/du}{dg/du}$$

simply by algebraic manipulation of the infinitesimals. (If we want to interpret these quantities as derivatives, then our notational convention is that they stand for the standard parts of the quotients of the infinitesimals, in which case the equality is only for the standard parts.) This equality holds not just in the limit but everywhere that the functions are differentiable. The expression on the left is the thing whose limit we're trying to prove equals lim $f/g$. The right-hand side is equal to lim $f/g$by the previously established form of l'Hˆopital's rule.

Problem 3.9 (Page 82)

By the definition of continuity in terms of infinitesimals, the function is continuous, because an infinitesimal change dxleads to a change d$y$= $a$ dxi n the output of the function which is likewise infinitesimals. (This depends on the fact that $a$is assumed to be real, which implies that it is finite.)

Continuity in terms of the Weierstrass limit holds because we can take

$\delta$= /$a$.

## Solutions for chapter 4



Problem 4.1 (Page 97):

a := 0;

b := 1;

```
H := 1000;

dt := (b-a)/H;

sum := 0;

t := a;

While (t<=b) [

sum := N(sum+Exp(x^2)*dt);

t := N(t+dt);

]; Echo(sum);
```

The result is 1.46.

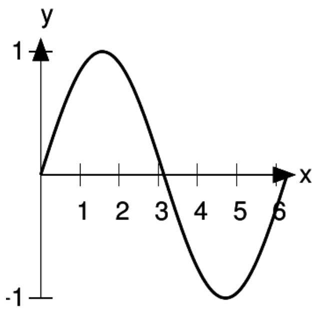


**Fig. 9.19: h / Problem 2.**

Problem 4.2 (Page 97):

The derivative of the cosine is minus the sine, so to get a function whose derivative is the sine, we need minus the cosine.

$$\int_0^{2\pi} sin x dx$$

$$\begin{aligned} &= (-cos x) \vert_0^{2\pi} \\ &= (-cos 2\pi) - (-cos 0) \\ &= (-1) - (-1) \\ &= 0 \end{aligned}$$

As shown in figure h, the graph has equal amounts of area above and below the *x*axis. The area below the axis counts as negative area, so the total is zero.

Problem 4.3 (Page 98):

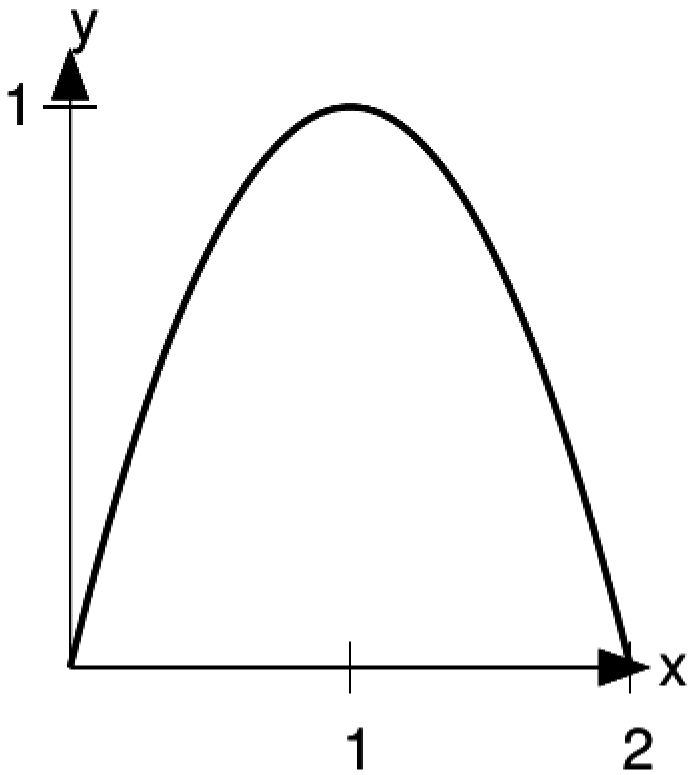


**Fig. 9.20: i / Problem 3.**

The rectangular area of the graph is 2, and the area under the curve fills a little more than half of that, so let's guess 1.4.

$$\begin{aligned}\int_0^2 -x^2 + 2x &= (-\frac{1}{3}x^3 + x^2) \mid_0^2 \\ &= (-8/3 + 4) - (0) \\ &= 4/3\end{aligned}$$

This is roughly what we were expecting from our visual estimate.

Problem 4.4 (Page 98):

Over this interval, the value of the sin function varies from 0 to 1, and it spends more time above 1/2 than below it, so we expect the average to be somewhat greater than 1/2. The exact result is

$$\begin{aligned}\bar{sin} &= \frac{1}{\pi - 0}\int_0^\pi sinx dx \\ &= \frac{1}{\pi}(-cosx) \mid_0^\pi \\ &= \frac{1}{\pi}[-cos\pi - (-cos0)] \\ &= \frac{2}{\pi}\end{aligned}$$

which is, as expected, somewhat more than 1/2.

Problem 4.5 (Page 98):

Consider a function *y*(*x*) defined on the interval from *x*= 0 to 2 like this:

$$y(x) = \begin{Bmatrix} -1 & if 0 \leq x \leq 1 \\ 1 & if 1 < x \leq 2 \end{Bmatrix}$$

The mean value of *y*is zero, but *y* never equals zero.

Problem 4.6 (Page 98):

Let $\dot{x}$ be defined as

$$\dot{x}(t) = \begin{Bmatrix} 0 & if t < 0 \\ 1 & if t \geq 0 \end{Bmatrix}$$

Integrating this function up to *t* gives

$$x(t) = \begin{Bmatrix} 0 & if t \leq 0 \\ t & if t \geq 0 \end{Bmatrix}$$

The derivative of *x* at *t*= 0 is undefined, and therefore integration followed by differentiation doesn't recover the original function $\dot{x}$ .

Problem 4.8 (Page 99):

First we put the integrand into the more familiar and convenient form $cx^p$, whose integral is$(c/(p=1))x^{p+1} \cdot \sqrt{bx\sqrt{x}} = b^{1/2}x^{3/4}$Applying the general rule, the result is $(4/7)b^{1/2}x^{7/4}$ .

Problem 4.11 (Page 100):

The claim is false for indefinite integrals, since indefinite integrals can have a constant of integration. So, for example, a possible indefinite integral of $x^2$ is $x^3/3+7$, which is neither even nor odd. The fundamental theorem doesn't even refer to indefinite integrals, which are simply *defined* through inverse differentiation.

Let's fix the claim by changing *g*to a definite integral, $g(x) = \int_0^x f(u)du$. The claim is now true. However, the proof still doesn't quite work. We've established that all odd functions have even derivatives, but we haven't ruled out possibilities such as functions that are neither even nor odd, but that have even derivatives.

## Solutions for chapter 5



Problem 5.16 (Page 120)

It's pretty trivial to generalize from *e* to *b*. If we write $b^x$ as $e^{x\,In\,b}$, then we can substitute *u* = *x* ln *b* and reduce the *b*= *e* case to *b* = *e*.

The generalization of the exponent of *x*from 2 to *a*is less straightforward. To do it with *a*= 2, we needed two integrations by parts, so clearly if we wanted to do a case with *a*= 37, we could do it with 37 integrations by parts. However, we would have no easy way to write down the complete answer without going through the whole tedious calculation. Furthermore, this is only going to work if *a*is a positive integer.

Problem 5.18 (Page 121)

The obvious substitution is $u = x^p$, which leads to the form $\int e^u u^{1/p-1} du$. If the exponent 1/*p*−1 equals a nonnegative integer *n*, then through *n*integrations by parts, we can reduce this to the form $\int e^x dx$. This requires *p*= 1, 1/2, 1/3, . . .

Problem 5.19 (Page 121)

This is a mess if attacked by brute force. The trick is to reexpress the function using partial fractions:

$$\frac{x^2+1}{x^3-x} = \frac{x^2+1}{2(x+1)} + \frac{x^2+1}{2(x-1)} - \frac{x^2+1}{x}$$

Writing *u*= *x*+ 1 and *v*= *x*−1, this becomes

$$u^{-1} + v^{-1} - x^{-1} + \dots,$$

where . . . represents terms that will not survive multiple differentiations. Since d*u*/d*x*= d*v*/d*x*= 1, the chain rule tells us that differentiation with respect to *u*or *v*is the same as differentiation with respect to *x*.

The result is 100!$(u^{-101} + v^{-101} - x^{-101})$, where the notation 100! means

1 ×2 ×...100.

## Solutions for chapter 6



The method of nding the inde nite integral is discussed in example 70 on p. 91 and problem 16 on p. 99. The result is $-(In2)^{-3}e^{-u}(-u^2-2u+2)$, where u = -x In 2. Plugging in the limits of integration, we obtain $2(In2)^{-3}$.

## Solutions for chapter 7



Problem 7.1 (Page 138)

We can define the sequence *f*(*n*) as converging to `if the following is true: for any real number, there exists an integer *N*such that for all *n* greater than *N*, the value of *f*lies within the range from $l - \varepsilon$ to $l + \epsilon$.

Problem 7.2 (Page 139)

(a) The convergence of the series is defined in terms of the convergence of its partial sums, which are 1, 0, 1, 0, . . . In the notation used in the definition given in the solution to problem 1 above, suppose we pick= 1⁄4. Then there is clearly no way to choose any numbers `and *N* that would satisfy the definition, for regardless of *N*, `would have to be both greater than 3⁄4 and less than 1⁄4 in order to agree with the zeroes and ones that occur beyond the *N*th member of the sequence.

(b) As remarked on page 106, the axioms of the real number system, such as associativity, only deal with finite sums, not infinite ones. To see that absurd conclusions result from attempting to apply them to infinite sums, consider that by the same type of argument we could group the sum as 1 + (−1 + 1) + (−1 + 1) + ..., which would equal 1.

Problem 7.3 (Page 139)

The quantity $x^n$ can be reexpressed as $e^{nInx}$, where ln *x*is negative by hypothesis. The integral of this exponential *with respect to n* is a similar exponential with a constant factor in front, and this converges as *n*approaches infinity.

Problem 7.4 (Page 139)

(a) Applying the integral test, we find that the integral of $1/x^2$ is -1/x, which converges as *x*approaches infinity, so the series converges as well.

(b) This is an alternating series whose terms approach zero, so it con-verges. However, the terms get small extremely slowly, so an extraordinarily large number of terms would be required in order to get any kind of decent approximation to the sum. In fact, it is impossible to carry out a straightforward numerical evaluation of this sum because it would require such an enormous number of terms that the rounding errors would overwhelm the result.

(c) This converges by the ratio test, because the ratio of successive terms approaches 0.

(d) Split the sum into two sums, one for the 1103 term and one for the 26390*k*. The ratio of the two factorials is always less than 44*k*, so discarding constant factors, the first sum is less than a geometric series with $x = (4/396)^4 < 1$, and must therefore converge. The second sum is less than a series of the form $kx^k$. This one also converges, by the integral test. (It has to be integrated with respect to *k*, not *x*, and the integration can be done by parts.) Since both separate sums converge, the entire sum converges. This bizarre-looking expression was formulated and shown to equal 1/*π*by the self-taught genius Srinivasa Ramanujan (1887-1920).

Problem 7.5 (Page 139)

E.g., $\sum_{n=0}^{\infty} sinn$ diverges, but the ratio test won't establish that, because the limit $lim_{n\to\infty}(n+1)/sin(n)|$ does not exist.

Problem 7.14 (Page 142)

The $n$ th term $a_n$ can be rewritten as 2/[$n$($n$+1)], and using partial fractions this can be changed into 2/$n$−2/($n$+ 1). Let the partial sums be $s_n = \sum_1^n a_n$. For insight, let's write out $s3$ :

$$s3 = (\frac{2}{1} - \frac{2}{2}) + (\frac{2}{2} - \frac{2}{3}) + \left(\frac{2}{3} - \frac{2}{4}\right)$$

This is called a telescoping series. The second part of one term cancels out with the first part of the next. Therefore we have

$$s^3 = \frac{2}{1} - \frac{2}{4}$$

and in general

$$s_n = \frac{2}{1} - \frac{2}{n+1}$$

Letting $n\to\infty$, we find that the series sums to 2.

Problem 7.17 (Page 143)

Yes, it converges. To see this, consider that its graph consists of a series of peaks and valleys, each of which is narrower than the last and therefore has less area. In fact, the width of these humps approaches zero, so that the area approaches zero. This means that the integral can be represented as a decreasing, alternating series that approaches zero, which must converge.

Problem 7.13 (Page 142)

There are certainly some special values of $x$ for which it does converge, such as 0 and $\pi$. For a general value of $x$, however, things become more complicated. Let the $n$th term be given by the function $t(n)$. $|t|$converges to a limit, since the first application of the sine function brings us into the range $0 \le |t| \le 1$, and from then on, $|t|$is decreasing and bounded below by 0. It can't approach a nonzero limit, for given such a limit $t*$, there would always be values of $t$ slightly greater than $t*$such that sin $t$was less than $t*$. Therefore the terms in the sum approach zero. This is necessary but not sufficient for the series to converge.

Once $t$ gets small enough, we can approximate the sine using a Taylor series. Approximating the discrete function $t$by a continuous one, we have $dt/dn \approx -(1/6)t^3$, which can be rewritten as $t^{-3}dt \approx (1/6)dn$. This is known as separation of variables. Integrating, we find that at large values of $n$, where the constant of integration becomes negligible, $t \approx \pm\sqrt{3/n}$. The sum diverges by the integral test. Therefore the sum diverges for all values of $x$except for multiples of $\pi$, which cause $t$ to hit zero immediately without passing through the region where the Taylor series is a good approximation.

## Problem 7.20 (Page 144)

Our first impression is that it must converge, since the $2^{-n}$ factor shrinks much more rapidly than the $n^2$ factor. To prove this rigorously, we can apply the integral test. The relevant improper integral was carried out in problem 4 on p. 104.

Finding the sum is far more difficult, and there is no obvious technique that is guaranteed to work. However, the integral test suggests an ap- proach that does lead to a solution. The fact that the *indefinite* integral can be evaluated suggests that perhaps the partial sum

$$S_n = \sum_{j=0}^{n} j^2 2^{-j}$$

can also be evaluated. Furthermore, the fact that the integral was of the form $2^{-x}P(x)$, for some polynomial *x*, suggests that perhaps $S_n$ is of the same form. Based on this conjecture, we try to determine the unknown coefficients in $P(n) = an^2 + bn + c$

$$S_n - S_{n-1} = n^2 2^{-n}$$
$$n^2 2^{-n} = 2^{-n}[-an^2 + (4a - b)n - 2a + 2b - c]$$

Solving for *a*, *b*, and *c* results in *P*(*n*) = $-n^2$ −4*n*−6. This gives the correct value for the difference $S_n - S_{n-1}$, but doesn't give $S_n = 0$ as it should. But this is easy to fix simply by changing the form of our conjectured partial sum slightly to $S_n = 2^{-n}P(n) + k$, where *k*= 6. Evaluating $lim_{n\to\infty} S_n$, we get 6.

## Problem 7.21 (Page 144)

The function $cos^2$ averages to 1/2, so we might naively expect that cos*n*would average to about $2^{-n/2}$, in which case the sum would converge for any value of *p* whatsoever. But the average is misleading, because there are some "lucky" values of *n*for which cos2 *n*≈1, and these will have a disproportionate effect on the sum. We know by the integral test that $\sum 1/n$ diverges, but $\sum 1/n^2$ converges, so clearly if *p*≥2, then even these occasional "lucky" terms will not cause divergence.

What about *p*= 1? Suppose we have some value of *n* for which $cos^2 n = 1 - \epsilon$, where is some small number. If this is to happen, then we must have *n*= *kπ*+ *δ*, where *k*is an integer and *δ*is small, so that $cos^2 n \approx 1 - \delta^2$, i.e.,$\epsilon \approx \delta^2$. This occurs with a probability proportional to *δ*, and the resulting contribution to the sum is about $(1 - \delta^{2n}/n)$, which by the binomial theorem is roughly of order of 1/n if $n\delta^2 \sim 1$. This happens with probabilityl $\sim n^{-1/2}$, so the expected value of the *n*th term is $\sim n^{-3/2}$. Since $\sum n^{-3/2}$ converges by the integral test, this suggests, but does not prove rigorously, that we also get convergence for

*p*= 1.

A similar argument suggests that the sum diverges for *p*= 0.

# Answers to self-checks for chapter 8



Problem 8.9 (Page 154)

First we rewrite the integrand as

$$\frac{1}{4}(e^{ix}+e^{-ix})(e^{2ix}+2^{-2ix})$$
$$=\frac{1}{4}(e^{3ix}+e^{-ix}+e^{ix}+e^{-ix})$$

The indefinite integral is

$$\frac{1}{12i}(e^{3ix}-e^{-3ix})+\frac{1}{4i}(e^{ix}-e^{-ix})$$

Evaluating this at 0 gives 0, while at =2 we nd 1/3. The result is 1/3.

Problem 8.8 (Page 154)

$$sin(a+b)=(e^{i(a+b)}-e^{-i(a+b)})/2i=(e^{ia}e^{ib}-e^{-ia}e^{-ib})/2i$$
$$=[(cosa+isina)(cosb+isinb)-(cosa-isina)(cosb-isinb)]/2i$$
$$=[(cosa+isina)(cosb+isinb)-(cosa-isina)(cosb-isinb)]/2i$$
$$=cosasinb+isinacosb$$

By a similar computation, we nd cos(a + b) = cos a cos b - sin a sin b.

Problem 8.10 (Page 155)

If $z^3=1$, then we know that |*z*|= 1, since cubing *z*cubes its magnitude. Cubing *z*triples its argument, so the argument of *z*must be a number that, when tripled, is equivalent to an angle of zero. There are three possibilities: 0 ×3 = 0, (2*π*/3) ×3 = 2*π*, and (4*π*/3)×3 = 4*π*. (Other possibilities, such as (32*π*/3), are equivalent to one of these.) The solutions are:

$$z=1,e^{2\pi i/3},e^{4\pi i/3}$$

Problem 8.11 (Page 155)

We can think of this as a polynomial in *x*or a polynomial in *y*— their roles are symmetric. Let's call *x*the variable. By the fundamental theorem of algebra, it must be possible to factor it into a product of three linear factors, if the coefficients are allowed to be complex. Each of these factors causes the product to be zero for a certain value of *x*. But the condition for the expression to be zero is *x*3 = *y*3 , which basically means that the ratio of *x*to *y*must be a third root of 1. The problem, then, boils down to finding the three third roots of 1, as in problem 10. Using the result of that problem, we find that there are zeroes when *x/y* equals 1, $e^{2\pi i/3}$, and $e^{4\pi i/3}$. This tells us that the factorization is $(x-y)(x-e^{2\pi i/3}y)(x-e^{4\pi i/3}y)$.

The second part of the problem asks us to factorize as much as possible using real coefficients. Our only hope of doing this is to multiply out the two factors that involve complex coefficients, and see if they produce something real. In fact, we can anticipate that it will work, because the coefficients are complex conjugates of one another, and when a quadratic has two complex roots, they are conjugates. The result is $(x-y)(x^2+xy+y^2)$.

Problem 8.14 (Page 155)

Applying the differential equation to the form suggested gives $abx^{b-1} = a^{b+1}x^{b^2}$. The exponents must be equal on both sides, so *b* must be a solution of $b^2 - b + 1$. The solutions are$b = (1 \pm \sqrt{3}i)/2$. For a more detailed discussion of this cute problem, see mathoverflow.net/questions/111066.

Problem 8.15 (Page 156)

(a) Let *m*= 10, 000. We know that integrals of this form can be done, at least in theory, using partial fractions. The ten thousand roots of the polynomial will be ten thousand points evenly spaced around the unit circle in the complex plane. They can be expressed as $r^k = e^{2\pi k/m}$ for *k*= 0 to *m*−1. Since all the roots are unequal, the partial-fraction form of the integrand contains only terms of the form $A_k/(x-r_k)$. Integrating, we would get a sum of ten thousand terms of the form $A_k In(x-r_k)$ .

(b) I tried inputting the integral into three different pieces of symbolic math software: the open-source packages Yacas and Maxima, and the web-based interface to Wolfram's proprietary Mathematica software at integrals.com. Maxima gave a partially integrated result after a couple of minutes of computation. Yacas crashed. Mathematica's web interface timed out and suggested buying a stand-alone copy of Mathematica. All three programs probably embarked on the computation of the *Ak* by attempting to solve 10,000 equations in the 10,000 unknowns *Ak*, and then ran out of resources (either memory or CPU time).

(c) The expressions look nicer if we let $w = e^{2\pi/m}$, so that $r_k = w^k$. The residue method gives

$$\frac{1}{x^m - 1} = \sum \frac{1}{(x - w^k m w^{k(m-1)})}$$

Integration gives

$$\int \frac{dx}{x^m - 1} = \sum \frac{1}{m w^{k(m-1)}} In(x - w^k)$$

(Thanks to math.stackexchage.com user zulon for suggesting the residue mathod, and to Robert Israel for pointing out that for |*x*|<1 this can also be expressed as a hypergeometric function:

$$(-x)_2F_1(\frac{1}{m}, 1; 1 + \frac{1}{m}; x^m)$$

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# References and Further Reading

## Further Reading



The amount of high-quality material on elementary calculus available for free online these days is an embarrassment of riches, so most of my suggestions for reading are online. I'll refer to books in this section only by the surname of the first author; the references section below tells you where to find the book online or in print.

The reader who wants to learn more about the hyperreal system might want to start with Stroyan and the Mathforum.org article. For more depth, one could next read the relevant parts of Keisler. The standard (difficult) treatise on the subject is Robinson.

Given sufficient ingenuity, it's possible to develop a surprisingly large amount of the machinery of calculus without using limits *or*infinitesimals. Two examples of such treatments that are freely available online are Marsden and Livshits. Marsden gives a geometrical definition of the derivative similar to the one used in ch. 1 of this book, but in my opinion his efforts to develop a sufficient body of techniques without limits or infinitesimals end up bogging down in complicated formulations that have the same flavor as the Weierstrass definition of the limit and are just as complicated. Livshits treats differentiation of rational functions as division of functions.

Tall gives an interesting construction of a number system that is smaller than the hyperreals, but easier to construct explicitly, and sufficient to handle calculus involving analytic functions.

## References



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www.math.wisc.edu/~keisler/calc.html

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# Reference

## E.1 Review

### Algebra



Quadratic equation:

The solutions of $ax^2 + bx + c = 0$

are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Logarithms and exponentials:

$$ln(ab) = lna + lnb$$
$$ln(ab) = lna + lnb$$
$$e^{a+b} = e^a e^b$$
$$Ine^x = e^{Inx} = x$$
$$In^{a^b} = bIna$$

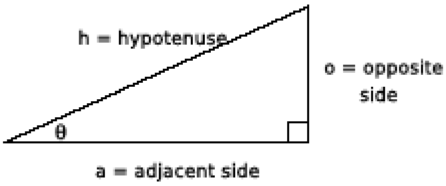
### Geometry, area, and volume



| | |
|---|---|
| area of a triangle of base *b*and height *h* | $= \frac{1}{2}bh$ |
| circumference of a circle of radius *r* | $= 2\pi r$ |
| area of a circle of radius r | $= \pi r^2$ |
| surface area of a sphere of radius *r* | $= 4\pi r^2$ |
| volume of a sphere of radius *r* | $= \frac{4}{3}\pi r^3$ |

## Trigonometry with a right triangle

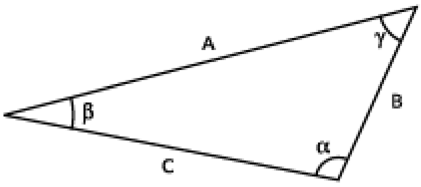





$sin\theta = o/h \quad cos\theta = a/h \quad tan\theta = o/a$
Pythagorean theorem: $h^2 = a^2 + 0^2$

## Trigonometry with any triangle






Law of Sines:

$$\frac{sin\alpha}{A} = \frac{sin\beta}{B} = \frac{sin\gamma}{C}$$

Law of Cosines:

$$C^2 = A^2 + B^2 - 2ABcos\gamma$$

# E.2 Hyperbolic functions



$$sinhx = \frac{e^x - e^{-x}}{2}$$
$$coshx = \frac{e^x + e^{-x}}{2}$$
$$tanhx = \frac{sinhx}{coshx}$$

## E.3 Calculus



Let *f* and *g* be functions of *x*, and let *c* be a constant.

Linearity of the derivative:

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

### Rules for differentiation



The chain rule:

$$\frac{d}{dx}f(g(x)) = f(g(x)g'(x)$$

Derivatives of products and quo-

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + \frac{dg}{dx}f$$

$$\frac{d}{dx}(\frac{f}{g}) = \frac{f'}{g} - \frac{fg'}{g^2}$$

### Integral calculus



The fundamental theorem of calculus:

$$\int \frac{df}{dx} = f$$

Linearity of the integral:

$$\int cf(x)dx = c\int f(x)dx$$

$$\int [f(x)+g(x)] = c\int f(x)dx + \int g(x)dx$$

Integration by parts:

$$\int f dg = fg - \int g df$$

## Table of integrals



$$\int x^m dx = \frac{1}{m+1}x^{m+1} + c, m \neq 1$$

$$\int \frac{dx}{x} = In \mid x \mid +c$$

$$\int sinx dx = -cosx + c$$

$$\int cosx dx = sinx + c$$

$$\int e^x dx = e^x + c$$

$$\int Inx dx = xInx - x + c$$

$$\int \frac{dx}{1+x^2} = tan^{-1}x + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1}x + c$$

$$\int coshx dx = sinhx + c$$

$$\int sinhx dx = coshx + c$$

$$\int tanx dx = -In \mid cosx \mid +c$$

$$\int cotx dx = In \mid sinx \mid +c$$

$$\int secx dx = In \mid secx + tanx \mid +c$$

$$\int sec^2 x dx = tanx + c$$

$$\int csc^2 x dx = -cotx + c$$